# Suplement of "Multiscale modeling of heat and mass transfer in dry snow: influence of the condensation coefficient and comparison with experiments"

Lisa Bouvet[1,2], Neige Calonne[1], Frédéric Flin[1], and Christian Geindreau[2]

[1]Univ. Grenoble Alpes, Université de Toulouse, Météo-France, CNRS, CNRM, Centre d'Études de la Neige, Grenoble, France

[2]Université Grenoble Alpes, CNRS, Grenoble INP, 3SR, Grenoble, France

**Correspondence:** Lisa Bouvet (lisa.bouvet@univ-grenoble-alpes.fr)

### S.1 Case A

Taking into account of the order of magnitude of the dimensionless numbers, $\left[\mathrm{F}_i^T\right] = \mathcal{O}\left(\left[\mathrm{F}_a^T\right]\right) = \mathcal{O}\left(\left[\mathrm{F}_a^\rho\right]\right) = \mathcal{O}(\varepsilon^2)$, $[\mathrm{K}] = \mathcal{O}\left(1\right), [\mathrm{H}] = \mathcal{O}\left(\varepsilon^2\right), [\mathrm{W_R}] = \mathcal{O}\left(\varepsilon^2\right)$, the dimensionless microscopic description (13)-(18) becomes:

$$\varepsilon^2 \rho_i^* C_i^* \frac{\partial T_i^*}{\partial t^*} - \mathrm{div}^*(k_i^* \mathbf{grad}^* T_i^*) = 0 \quad \text{in } \Omega_i \tag{A.1}$$

$$\varepsilon^2 \rho_a^* C_a^* \frac{\partial T_a^*}{\partial t^*} - \mathrm{div}^*(k_a^* \mathbf{grad} T_a^*) = 0 \quad \text{in } \Omega_a \tag{A.2}$$

$$\varepsilon^2 \frac{\partial \rho_v^*}{\partial t^*} - \mathrm{div}^*(D_v^* \mathbf{grad}^* \rho_v^*) = 0 \quad \text{in } \Omega_a \tag{A.3}$$

$$T_i^* = T_a^* \quad \text{on } \Gamma \tag{A.4}$$

$$k_i^* \mathbf{grad}^* T_i^* \cdot \mathbf{n_i} - k_a^* \mathbf{grad}^* T_a^* \cdot \mathbf{n_i} = \varepsilon^2 L_{sg}^* \mathbf{w}^* \cdot \mathbf{n_i} \quad \text{on } \Gamma \tag{A.5}$$

$$D_v^* \mathbf{grad}^* \rho_v^* \cdot \mathbf{n_i} = \varepsilon^2 \rho_i^* \mathbf{w}^* \cdot \mathbf{n_i} \quad \text{on } \Gamma. \tag{A.6}$$

This set of equations is completed by the Hertz-Knudsen equation (10) and the Clausius Clapeyron's law (9) expressed in dimensionless form as:

$$w_n^* = \mathbf{w}^* \cdot \mathbf{n_i} = \frac{\alpha^*}{\rho_i^*} w_\mathrm{k}^* \left[\rho_v^* - \rho_{vs}^*(T_a^*)\right] \quad \text{on } \Gamma \tag{A.7}$$

$$\rho_{vs}^*(T_a^*) = \rho_{vs}^{\mathrm{ref}*}(T^{\mathrm{ref}*}) \exp\left[\frac{L_{sg}^* m^*}{\rho_i^* k^*}\left(\frac{1}{T^{\mathrm{ref}*}} - \frac{1}{T_a^*}\right)\right] \tag{A.8}$$

### S.1.1 Heat transfer

Introducing asymptotic expansions for $T_i^*$ and $T_a^*$ in the relations (A.1), (A.2), (A.4), (A.5) gives at the lowest order:

$$\mathrm{div}_{y^*}(k_i^* \mathbf{grad}_{y^*} T_i^{*(0)}) = 0 \quad \text{in } \Omega_i \tag{A.9}$$

$$\mathrm{div}_{y^*}(k_a^* \mathbf{grad}_{y^*} T_a^{*(0)}) = 0 \quad \text{in } \Omega_a \tag{A.10}$$

$$T_i^{*(0)} = T_a^{*(0)} \quad \text{on } \Gamma \tag{A.11}$$

$$(k_i^* \mathbf{grad}_{y^*}^* T_i^{*(0)} - k_a^* \mathbf{grad}_{y^*}^* T_a^{*(0)}) \cdot \mathbf{n_i} = 0 \quad \text{on } \Gamma \tag{A.12}$$

where the unknowns $T_i^{*(0)}(\mathbf{x}^*, \mathbf{y}^*, t)$ and $T_a^{*(0)}(\mathbf{x}^*, \mathbf{y}^*, t)$ are $\mathbf{y}^*$-periodic. It can be shown that the obvious solution of the above boundary value problem is given by:

$$T_i^{*(0)} = T_a^{*(0)} = T^{*(0)}(\mathbf{x}^*, t). \tag{A.13}$$

At the first order, the temperature is independent of the microscopic dimensionless variable $\mathbf{y}^*$, i.e. we have only one temperature field. Taking into account of these results, equations (A.1), (A.2), (A.4), and (A.5) of order $\varepsilon$ give the following second-order problem:

$$\mathrm{div}_{y^*}(k_i^*(\mathbf{grad}_{y^*} T_i^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)})) = 0 \quad \text{in } \Omega_i \tag{A.14}$$

$$\mathrm{div}_{y^*}(k_a^*(\mathbf{grad}_{y^*} T_a^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)})) = 0 \quad \text{in } \Omega_a \tag{A.15}$$

$$T_i^{*(1)} = T_a^{*(1)} \quad \text{on } \Gamma \tag{A.16}$$

$$(k_i^*(\mathbf{grad}_{y^*} T_i^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)}) - k_a^*(\mathbf{grad}_{y^*} T_a^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)})) \cdot \mathbf{n_i} = 0 \quad \text{on } \Gamma \tag{A.17}$$

where the unknowns $T_i^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t)$ and $T_a^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t)$ are $\mathbf{y}^*$-periodic and the macroscopic gradient $\mathbf{grad}_{x^*} T^{*(0)}$ is given. The solution of the above boundary value problem appears as a linear function of the macroscopic gradient, modulo an arbitrary function $\widetilde{T}^{*(1)}(\mathbf{x}^*, t)$:

$$T_i^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t) = \mathbf{t}_i^*(\mathbf{y}^*) \cdot \mathbf{grad}_{x^*} T^{*(0)} + \widetilde{T}_i^{*(1)} \tag{A.18}$$

$$T_a^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t) = \mathbf{t}_a^*(\mathbf{y}^*) \cdot \mathbf{grad}_{x^*} T^{*(0)} + \widetilde{T}_a^{*(1)} \tag{A.19}$$

where $\mathbf{t}_i^*(\mathbf{y}^*)$ and $\mathbf{t}_a^*(\mathbf{y}^*)$ are two periodic vectors which characterize the fluctuation of temperature in both phases at the pore scale. Introducing (A.18) and (A.19) in the set (A.14)-(A.17), these two vectors are solution of the following boundary value problem, expressed in a compact form as:

$$\mathrm{div}_{y^*}(k_i^*(\mathbf{grad}_{y^*}\mathbf{t}_i^* + \mathbf{I})) = 0 \quad \text{in } \Omega_i \tag{A.20}$$

$$\mathrm{div}_{y^*}(k_a^*(\mathbf{grad}_{y^*}\mathbf{t}_a^* + \mathbf{I})) = 0 \quad \text{in } \Omega_a \tag{A.21}$$

$$\mathbf{t}_i^* = \mathbf{t}_a^* \quad \text{on } \Gamma \tag{A.22}$$

$$(k_i^*(\mathbf{grad}_{y^*}\mathbf{t}_i^* + \mathbf{I}) - k_a^*(\mathbf{grad}_{y^*}\mathbf{t}_a^* + \mathbf{I})) \cdot \mathbf{n_i} = 0 \quad \text{on } \Gamma \tag{A.23}$$

$$\frac{1}{|\Omega|}\int_\Omega (\mathbf{t}_\mathrm{a}^* + \mathbf{t}_\mathrm{i}^*)\mathrm{d}\Omega = \mathbf{0} \tag{A.24}$$

This latter equation is introduced to ensure the uniqueness of the solution. Finally, the third order problem is given by the Eq. (A.1), (A.2), (A.4), and (A.5) of order $\varepsilon^2$:

$$\rho_i^* C_i^* \frac{\partial T^{*(0)}}{\partial t^*} - \mathrm{div}_{y^*}(k_i^*(\mathbf{grad}_{y^*} T_i^{*(2)} + \mathbf{grad}_{x^*} T_i^{*(1)})) - \mathrm{div}_{x^*}(k_i^*(\mathbf{grad}_{y^*} T_i^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)})) = 0 \quad \text{in } \Omega_i \tag{A.25}$$

$$\rho_a^* C_a^* \frac{\partial T^{*(0)}}{\partial t^*} - \mathrm{div}_{y^*}(k_a^*(\mathbf{grad}_{y^*} T_a^{*(2)} + \mathbf{grad}_{x^*} T_a^{*(1)})) - \mathrm{div}_{x^*}(k_a^*(\mathbf{grad}_{y^*} T_a^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)})) = 0 \quad \text{in } \Omega_a \tag{A.26}$$

$$T_i^{*(2)} = T_a^{*(2)} \quad \text{on } \Gamma \tag{A.27}$$

$$(k_i^*(\mathbf{grad}_{y^*} T_i^{*(2)} + \mathbf{grad}_{x^*} T_i^{*(1)}) - k_a^*(\mathbf{grad}_{y^*} T_a^{*(2)} + \mathbf{grad}_{x^*} T_a^{*(1)})) \cdot \mathbf{n_i} = w_n^{*(0)} \quad \text{on } \Gamma \tag{A.28}$$

where the unknowns $T_i^{*(2)}(\mathbf{x}^*, \mathbf{y}^*, t)$ and $T_a^{*(2)}(\mathbf{x}^*, \mathbf{y}^*, t)$ are $\mathbf{y}^*$-periodic and $w_n^{*(0)}$ is the normal interface velocity due to the sublimation-deposition process given, at the zero order, by the Hertz-Knudsen equation (A.7) and the Clausius Clapeyron's law (A.8).

### S.1.2 Water vapor transfer

Introducing asymptotic expansions for $\rho_v^*$ in the relations (A.3) and (A.6) gives at the lowest order:

$$\mathrm{div}_{y^*}(D_v^* \mathbf{grad}_{y^*} \rho_v^{*(0)}) = 0 \quad \text{in } \Omega_a \tag{A.29}$$

$$D_v^* \mathbf{grad}_{y^*} \rho_v^{*(0)} \cdot \mathbf{n_i} = 0 \quad \text{on } \Gamma. \tag{A.30}$$

where the unknown $\rho_v^{*(0)}(\mathbf{x}^*, \mathbf{y}^*, t)$ is $\mathbf{y}^*$-periodic. It can be shown (Auriault et al., 2009) that the solution of the above boundary value problem is given by:

$$\rho_v^{*(0)} = \rho_v^{*(0)}(\mathbf{x}^*, t). \tag{A.31}$$

At the first order, the water vapor density is independent of the microscopic dimensionless variable $\mathbf{y}^*$. Taking into account of these results, the second-order problem is given by Eq. (A.3) and (A.6) of order $\varepsilon$, which are:

$$\mathrm{div}_{y^*}(D_v^*(\mathbf{grad}_{y^*} \rho_v^{*(1)} + \mathbf{grad}_{x^*} \rho_v^{*(0)})) = 0 \quad \text{in } \Omega_a \tag{A.32}$$

$$D_v^*(\mathbf{grad}_{y^*} \rho_v^{*(1)} + \mathbf{grad}_{x^*} \rho_v^{*(0)}) \cdot \mathbf{n_i} = 0 \quad \text{on } \Gamma. \tag{A.33}$$

where the unknown $\rho_v^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t)$ is $\mathbf{y}^*$-periodic and the macroscopic gradient $\mathbf{grad}_{x^*} \rho_v^{*(0)}$ is given. The solution of the above boundary value problem appears as a linear function of the macroscopic gradient, modulo an arbitrary function $\widetilde{\rho}_v^{*(1)}(\mathbf{x}^*, t)$ (Auriault et al., 2009):

$$\rho_v^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t) = \mathbf{g}_v^*(\mathbf{y}^*) \cdot \mathbf{grad}_{x^*} \rho_v^{*(0)} + \widetilde{\rho}_v^{*(1)}(\mathbf{x}^*, t) \tag{A.34}$$

where $\mathbf{g}_v^*(\mathbf{y}^*)$ is a periodic vector which characterizes the fluctuation of water vapor density in the air phase at the pore scale. Introducing (A.34) in the set (A.32)-(A.33), this vector is solution of the following boundary value problem, expressed in a compact form:

$$\mathrm{div}_{y^*}(D_v^*(\mathbf{grad}_{y^*} \mathbf{g}_v^* + \mathbf{I})) = 0 \quad \text{in } \Omega_a \tag{A.35}$$

$$D_v^*(\mathbf{grad}_{y^*}\mathbf{g}_v^* + \mathbf{I}) \cdot \mathbf{n_i} = 0 \quad \text{on } \Gamma \tag{A.36}$$

$$\frac{1}{|\Omega|}\int_{\Omega_a} \mathbf{g}_v^* \mathrm{d}\Omega = \mathbf{0} \tag{A.37}$$

This latter equation is introduced to ensure the uniqueness of the solution. Finally, the third order problem is given by Eq. (A.3) and (A.6) of order $\varepsilon^2$:

$$\frac{\partial \rho_v^{*(0)}}{\partial t^*} - \mathrm{div}_{y^*}(D_v^*(\mathbf{grad}_{y^*}\rho_v^{*(2)} + \mathbf{grad}_{x^*}\rho_v^{*(1)})) - \mathrm{div}_{x^*}(D_v^*(\mathbf{grad}_{y^*}\rho_v^{*(1)} + \mathbf{grad}_{x^*}\rho_v^{*(0)})) = 0 \quad \text{in } \Omega_a \tag{A.38}$$

$$D_v^*(\mathbf{grad}_{y^*}\rho_v^{*(2)} + \mathbf{grad}_{x^*}\rho_v^{*(1)}) \cdot \mathbf{n_i} = w_n^{*(0)} \quad \text{on } \Gamma \tag{A.39}$$

where the unknown $\rho_v^{*(2)}(\mathbf{x}^*, \mathbf{y}^*, t)$ is $\mathbf{y}^*$-periodic and $w_n^{*(0)}$ is the normal interface velocity due to the sublimation/deposition process given, at the zero order, by the Hertz-Knudsen equation (A.7) and the Clausius Clapeyron's law (A.8). Taking into account the above results, we have:

$$\rho_{vs}^*(T_a^*) = \rho_{vs}^{\mathrm{ref}*}(T^{\mathrm{ref}*}) \exp\left[\frac{L_{sg}^* m^*}{\rho_i^* k^*}\left(\frac{1}{T^{\mathrm{ref}*}} - \frac{1}{T^{*(0)}}\right)\right]\left(1 + \varepsilon \frac{L_{sg}^* m^*}{\rho_i^* k^*}\frac{T_a^{*(1)}}{(T^{*(0)})^2} + ...\right) \tag{A.40}$$

This relation shows that the asymptotic development of the Clausius-Clapeyron's law is written:

$$\rho_{vs}^*(T_a^*) = \rho_{vs}^{*(0)}(\mathbf{x}^*, t) + \varepsilon \rho_{vs}^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t) + ... \tag{A.41}$$

where the first term $\rho_{vs}^{*(0)}$, which depends on $T^{*(0)}(\mathbf{x}^*, t)$ only, is defined as:

$$\rho_{vs}^{*(0)}(T^{*(0)}) = \rho_{vs}^{\mathrm{ref}*}(T^{\mathrm{ref}*}) \exp\left[\frac{L_{sg}^* m^*}{\rho_i^* k^*}\left(\frac{1}{T^{\mathrm{ref}*}} - \frac{1}{T^{*(0)}}\right)\right] \tag{A.42}$$

The relation (A.42) shows that the normal velocity $w_n^{*(0)}$ arising in the boundary condition (A.39) does not depend on $\mathbf{y}^*$. From (A.7), $w_n^{*(0)}$ is also written:

$$w_n^{*(0)} = \frac{\alpha^*}{\rho_i^*} w_\mathrm{k}^* \left[\rho_v^{*(0)} - \rho_{vs}^{*(0)}(T^{*(0)})\right] \tag{A.43}$$

### S.1.3 Macroscopic description

Integrating (A.25) over $\Omega_i$ and (A.26) over $\Omega_a$, and then using the divergence theorem, the periodicity condition, and the boundary conditions (A.28) leads to the first order dimensionless description:

$$(\rho C)^{\mathrm{eff}*}\frac{\partial T^{*(0)}}{\partial t^*} - \mathrm{div}_{x^*}(\mathbf{k}^{\mathrm{eff}*}\mathbf{grad}_{x^*}\, T^{*(0)}) = \mathrm{SSA_V} L_{sg}^* w_n^{*(0)} \tag{A.44}$$

where $\mathrm{SSA_V} = |\Gamma|/|\Omega|$ is the specific surface area and where $(\rho C)^{\mathrm{eff}*}$ and $\mathbf{k}^{\mathrm{eff}*}$ are the dimensionless effective thermal capacity and the effective dimensionless thermal conductivity, respectively, defined as:

$$(\rho C)^{\mathrm{eff}*} = (1-\phi)\rho_i^* C_i^* + \phi \rho_a^* C_a^* \tag{A.45}$$

$$\mathbf{k}^{\mathrm{eff}*} = \frac{1}{|\Omega|}\left(\int_{\Omega_\mathrm{a}} k_a^*(\mathbf{grad}_{y^*}\mathbf{t}_a^*(\mathbf{y}^*) + \mathbf{I})\mathrm{d}\Omega + \int_{\Omega_\mathrm{i}} k_i^*(\mathbf{grad}_{y^*}\mathbf{t}_i^*(\mathbf{y}^*) + \mathbf{I})\mathrm{d}\Omega\right) \tag{A.46}$$

where $\phi$ is the porosity. Consequently, integrating (A.38) over $\Omega_a$, and then using the divergence theorem, the periodicity condition, and the boundary conditions (A.39) leads to the first order dimensionless description:

$$\phi\frac{\partial \rho_v^{*(0)}}{\partial t} - \mathrm{div}_{x^*}(\mathbf{D}^{\mathrm{eff}*}\mathbf{grad}_{x^*}\rho_v^{*(0)}) = -\mathrm{SSA_V}\rho_i^* w_n^{*(0)} \tag{A.47}$$

where $\mathbf{D}^{\mathrm{eff}*}$ is the dimensionless effective diffusion tensor defined as:

$$\mathbf{D}^{\mathrm{eff}*} = \frac{1}{|\Omega|}\int_{\Omega_a} D_v^*(\mathbf{grad}_{y^*}\ \mathbf{g}_v^*(\mathbf{y}^*) + \mathbf{I})\mathrm{d}\,\Omega \tag{A.48}$$

## S.2 Case B1 and B2

### S.2.1 Case B1

Taking into account of the order of magnitude of the dimensionless numbers, $\left[\mathrm{F}_i^T\right] = \mathcal{O}\left(\left[\mathrm{F}_a^T\right]\right) = \mathcal{O}\left([\mathrm{F}_a^\rho]\right) = \mathcal{O}(\varepsilon^2)$, $[\mathrm{K}] = \mathcal{O}(1)\,, [\mathrm{H}] = \mathcal{O}(\varepsilon)\,, [\mathrm{W_R}] = \mathcal{O}(\varepsilon)$, the dimensionless microscopic description (13)-(18) becomes:

$$\varepsilon^2 \rho_i^* C_i^* \frac{\partial T_i^*}{\partial t^*} - \mathrm{div}^*(k_i^*\mathbf{grad}^* T_i^*) = 0 \quad \text{in } \Omega_i \tag{B1.1}$$

$$\varepsilon^2 \rho_a^* C_a^* \frac{\partial T_a^*}{\partial t^*} - \mathrm{div}^*(k_a^*\mathbf{grad} T_a^*) = 0 \quad \text{in } \Omega_a \tag{B1.2}$$

$$\varepsilon^2 \frac{\partial \rho_v^*}{\partial t^*} - \mathrm{div}^*(D_v^*\mathbf{grad}^*\rho_v^*) = 0 \quad \text{in } \Omega_a \tag{B1.3}$$

$$T_i^* = T_a^* \quad \text{on } \Gamma \tag{B1.4}$$

$$k_i^*\mathbf{grad}^* T_i^* \cdot \mathbf{n_i} - k_a^*\mathbf{grad}^* T_a^* \cdot \mathbf{n_i} = \varepsilon L_{sg}^* \mathbf{w}^* \cdot \mathbf{n_i} \quad \text{on } \Gamma \tag{B1.5}$$

$$D_v^*\mathbf{grad}^*\rho_v^* \cdot \mathbf{n_i} = \varepsilon \rho_i^* \mathbf{w}^* \cdot \mathbf{n_i} \quad \text{on } \Gamma. \tag{B1.6}$$

This set of equations is completed by the Hertz-Knudsen equation (10) and the Clausius Clapeyron's law (9) expressed in dimensionless form as:

$$w_n^* = \mathbf{w}^* \cdot \mathbf{n_i} = \frac{\alpha^*}{\rho_i^*} w_\mathrm{k}^* \left[\rho_v^* - \rho_{vs}^*(T_a^*)\right] \;\text{ on } \Gamma \tag{B1.7}$$

$$\rho_{vs}^*(T_a^*) = \rho_{vs}^{\mathrm{ref}*}(T^{\mathrm{ref}*}) \exp\left[\frac{L_{sg}^* m^*}{\rho_i^* k^*}\left(\frac{1}{T^{\mathrm{ref}*}} - \frac{1}{T_a^*}\right)\right] \tag{B1.8}$$

#### S.2.1.1 Heat transfer

Introducing asymptotic expansions for $T_i^*$ and $T_a^*$ in the relations (B1.1), (B1.2), (B1.4), and (B1.5) gives at the lowest order:

$$\mathrm{div}_{y^*}(k_i^* \mathbf{grad}_{y^*} T_i^{*(0)}) = 0 \quad \text{in } \Omega_i \tag{B1.9}$$

$$\mathrm{div}_{y^*}(k_a^* \mathbf{grad}_{y^*} T_a^{*(0)}) = 0 \quad \text{in } \Omega_a \tag{B1.10}$$

$$T_i^{*(0)} = T_a^{*(0)} \quad \text{on } \Gamma \tag{B1.11}$$

$$(k_i^* \mathbf{grad}_{y^*}^* T_i^{*(0)} - k_a^* \mathbf{grad}_{y^*}^* T_a^{*(0)}) \cdot \mathbf{n_i} = 0 \quad \text{on } \Gamma \tag{B1.12}$$

where the unknowns $T_i^{*(0)}(\mathbf{x}^*, \mathbf{y}^*, t)$ and $T_a^{*(0)}(\mathbf{x}^*, \mathbf{y}^*, t)$ are $\mathbf{y}^*$-periodic. It can be shown Auriault et al. (2009) that the obvious solution of the above boundary value problem is given by:

$$T_i^{*(0)} = T_a^{*(0)} = T^{*(0)}(\mathbf{x}^*, t). \tag{B1.13}$$

At the first order, the temperature is independent of the microscopic dimensionless variable $\mathbf{y}^*$, i.e. we have only one temperature field. Taking into account of these results, Eq. (B1.1), (B1.2), (B1.4), and (B1.5) of order $\varepsilon$ give the following second-order problem:

$$\mathrm{div}_{y^*}(k_i^*(\mathbf{grad}_{y^*} T_i^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)})) = 0 \quad \text{in } \Omega_i \tag{B1.14}$$

$$\mathrm{div}_{y^*}(k_a^*(\mathbf{grad}_{y^*} T_a^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)})) = 0 \quad \text{in } \Omega_a \tag{B1.15}$$

$$T_i^{*(1)} = T_a^{*(1)} \quad \text{on } \Gamma \tag{B1.16}$$

$$(k_i^*(\mathbf{grad}_{y^*} T_i^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)}) - k_a^*(\mathbf{grad}_{y^*} T_a^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)})) \cdot \mathbf{n_i} = L_{sg}^* w_n^{*(0)} \quad \text{on } \Gamma \tag{B1.17}$$

where the unknowns $T_i^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t)$ and $T_a^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t)$ are $\mathbf{y}^*$-periodic and the macroscopic gradient $\mathbf{grad}_{x^*} T^{*(0)}$ is given. Moreover, it can be shown that at the first order $w_n^{*(0)} = 0$ (see B1.37). As in the case A, the solution of the above boundary value problem appears as a linear function of the macroscopic gradient, modulo an arbitrary function $\widetilde{T}^{*(1)}(\mathbf{x}^*, t)$ Auriault et al. (2009):

$$T_i^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t) = \mathbf{t}_i^*(\mathbf{y}^*) \cdot \mathbf{grad}_{x^*} T^{*(0)} + \widetilde{T}_i^{*(1)} \tag{B1.18}$$

$$T_a^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t) = \mathbf{t}_a^*(\mathbf{y}^*) \cdot \mathbf{grad}_{x^*} T^{*(0)} + \widetilde{T}_a^{*(1)} \tag{B1.19}$$

where $\mathbf{t}_i^*(\mathbf{y}^*)$ and $\mathbf{t}_a^*(\mathbf{y}^*)$ are two periodic vectors which characterize the fluctuation of temperature in both phases at the pore scale. Introducing (B1.18) and (B1.19) in the set (B1.14)-(B1.17), these two vectors are solution of the following boundary value problem in a compact form:

$$\mathrm{div}_{y^*}(k_i^*(\mathbf{grad}_{y^*} \mathbf{t}_i^* + \mathbf{I})) = 0 \quad \text{in } \Omega_i \tag{B1.20}$$

$$\mathrm{div}_{y^*}(k_a^*(\mathbf{grad}_{y^*} \mathbf{t}_a^* + \mathbf{I})) = 0 \quad \text{in } \Omega_a \tag{B1.21}$$

$$\mathbf{t}_i^* = \mathbf{t}_a^* \quad \text{on } \Gamma \tag{B1.22}$$

$$(k_i^*(\mathbf{grad}_{y^*} \mathbf{t}_i^* + \mathbf{I}) - k_a^*(\mathbf{grad}_{y^*} \mathbf{t}_a^* + \mathbf{I})) \cdot \mathbf{n_i} = 0 \quad \text{on } \Gamma \tag{B1.23}$$

$$\frac{1}{|\Omega|} \int_{\Omega} (\mathbf{t}_\mathrm{a}^* + \mathbf{t}_\mathrm{i}^*) \mathrm{d}\Omega = \mathbf{0} \tag{B1.24}$$

This latter equation is introduced to ensure the uniqueness of the solution. Finally, the third order problem is given by Eq. (B1.1), (B1.2), (B1.4), and (B1.5) of order $\varepsilon^2$:

$$\rho_i^* C_i^* \frac{\partial T^{*(0)}}{\partial t^*} - \mathrm{div}_{y^*}(k_i^*(\mathbf{grad}_{y^*} T_i^{*(2)} + \mathbf{grad}_{x^*} T_i^{*(1)})) - \mathrm{div}_{x^*}(k_i^*(\mathbf{grad}_{y^*} T_i^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)})) = 0 \quad \text{in } \Omega_i \tag{B1.25}$$

$$\rho_a^* C_a^* \frac{\partial T^{*(0)}}{\partial t^*} - \mathrm{div}_{y^*}(k_a^*(\mathbf{grad}_{y^*} T_a^{*(2)} + \mathbf{grad}_{x^*} T_a^{*(1)})) - \mathrm{div}_{x^*}(k_a^*(\mathbf{grad}_{y^*} T_a^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)})) = 0 \quad \text{in } \Omega_a \tag{B1.26}$$

$$T_i^{*(2)} = T_a^{*(2)} \quad \text{on} \Gamma \tag{B1.27}$$

$$(k_i^*(\mathbf{grad}_{y^*} T_i^{*(2)} + \mathbf{grad}_{x^*} T_i^{*(1)}) - k_a^*(\mathbf{grad}_{y^*} T_a^{*(2)} + \mathbf{grad}_{x^*} T_a^{*(1)})) \cdot \mathbf{n_i} = L_{sg}^* w_n^{*(1)} \quad \text{on } \Gamma \tag{B1.28}$$

where the unknowns $T_i^{*(2)}(\mathbf{x}^*, \mathbf{y}^*, t)$ and $T_a^{*(2)}(\mathbf{x}^*, \mathbf{y}^*, t)$ are $\mathbf{y}^*$-periodic. Integrating (B1.25) over $\Omega_i$ and (B1.26) over $\Omega_a$, and then using the divergence theorem, the periodicity condition, and the boundary conditions (B1.28) leads to the first order dimensionless description:

$$(\rho C)^{\text{eff}*} \frac{\partial T^{*(0)}}{\partial t^*} - \mathrm{div}_{x^*}(\mathbf{k}^{\text{eff}*} \mathbf{grad}_{x^*} T^{*(0)}) = \int_\Gamma L_{sg}^* w_n^{*(1)} dS = -L_{sg}^* \dot{\phi} \tag{B1.29}$$

where $(\rho C)^{\text{eff}*}$ and $\mathbf{k}^{\text{eff}*}$ are the dimensionless effective thermal capacity and the effective dimensionless thermal conductivity respectively, defined, as in the Case A, by:

$$(\rho C)^{\text{eff}*} = (1-\phi)\rho_i^* C_i^* + \phi \rho_a^* C_a^* \tag{B1.30}$$

$$\mathbf{k}^{\text{eff}*} = \frac{1}{|\Omega|} \left( \int_{\Omega_\mathrm{a}} k_a^*(\mathbf{grad}_{y^*} \mathbf{t}_a^*(\mathbf{y}^*) + \mathbf{I}) \mathrm{d}\Omega + \int_{\Omega_\mathrm{i}} k_i^*(\mathbf{grad}_{y^*} \mathbf{t}_i^*(\mathbf{y}^*) + \mathbf{I}) \mathrm{d}\Omega \right) \tag{B1.31}$$

where $\phi$ is the porosity.

#### S.2.1.2 Water vapor transfer

Introducing asymptotic expansions for $\rho_v^*$ in the relations (B1.3) and (B1.6) gives at the lowest order:

$$\mathrm{div}_{y^*}(D_v^* \mathbf{grad}_{y^*} \rho_v^{*(0)}) = 0 \quad \text{in } \Omega_a \tag{B1.32}$$

$$D_v^* \mathbf{grad}_{y^*} \rho_v^{*(0)} \cdot \mathbf{n_i} = 0 \quad \text{on } \Gamma. \tag{B1.33}$$

where the unknown $\rho_v^{*(0)}(\mathbf{x}^*, \mathbf{y}^*, t)$ is $\mathbf{y}^*$-periodic. It can be shownAuriault et al. (2009) that the solution of the above boundary value problem is given by:

$$\rho_v^{*(0)} = \rho_v^{*(0)}(\mathbf{x}^*, t). \tag{B1.34}$$

At the first order, the water vapor density is independent of the microscopic dimensionless variable $\mathbf{y}^*$. Taking into account of these results, the second-order problem is given by Eq. (B1.3) and (B1.6) of order $\varepsilon$:

$$\mathrm{div}_{y^*}(D_v^*(\mathbf{grad}_{y^*}\rho_v^{*(1)} + \mathbf{grad}_{x^*}\rho_v^{*(0)})) = 0 \quad \text{in } \Omega_a \tag{B1.35}$$

$$D_v^*(\mathbf{grad}_{y^*}\rho_v^{*(1)} + \mathbf{grad}_{x^*}\rho_v^{*(0)}) \cdot \mathbf{n_i} = \alpha w_\mathrm{k}\left[\rho_v^{*(0)} - \rho_{vs}^{*(0)}(T^{*(0)})\right] \quad \text{on } \Gamma. \tag{B1.36}$$

where the unknown $\rho_v^{*(1)}(\mathbf{x}^*,\mathbf{y}^*,t)$ is $\mathbf{y}^*$-periodic. Consequently, integrating (B1.35) over $\Omega_a$, and then using the divergence theorem, the periodicity condition, the boundary conditions (B1.36) and the result (B1.34) leads to the first order dimensionless description:

$$\rho_v^{*(0)} = \rho_{vs}^{*(0)}(T^{*(0)}) \tag{B1.37}$$

Consequently, as in the Case A, the solution of the above boundary value problem (B1.35) - (B1.36) appears as a linear function of the macroscopic gradient $\mathbf{grad}_{x^*}\rho_{vs}^{*(0)}(T^{*(0)})$ modulo an arbitrary function $\widetilde{\rho}_v^{*(1)}(\mathbf{x}^*,t)$ :

$$\rho_v^{*(1)}(\mathbf{x}^*,\mathbf{y}^*,t) = \mathbf{g}_v^*(\mathbf{y}^*) \cdot \mathbf{grad}_{x^*}\rho_{vs}^{*(0)}(T^{*(0)}) + \widetilde{\rho}_v^{*(1)}(\mathbf{x}^*,t) \tag{B1.38}$$

where $\mathbf{g}_v^*(\mathbf{y}^*)$ is a periodic vector which characterizes the fluctuation of water vapor density in the air phase at the pore scale induced by the macroscopic gradient $\mathbf{grad}_{x^*}\rho_{vs}^{*(0)}(T^{*(0)})$. Introducing (B1.38) in the set (B1.35)-(B1.36), this vector is solution of the following boundary value problem in a compact form:

$$\mathrm{div}_{y^*}(D_v^*(\mathbf{grad}_{y^*}\mathbf{g}_v^* + \mathbf{I})) = 0 \quad \text{in } \Omega_a \tag{B1.39}$$

$$D_v^*(\mathbf{grad}_{y^*}\mathbf{g}_v^* + \mathbf{I}) \cdot \mathbf{n_i} = 0 \quad \text{on } \Gamma \tag{B1.40}$$

$$\frac{1}{|\Omega|}\int_{\Omega_a} \mathbf{g}_v^* \mathrm{d}\Omega = \mathbf{0} \tag{B1.41}$$

This latter equation is introduced to ensure the uniqueness of the solution. Finally, the third order problem is given by Eq. (B1.3) and (B1.6) of order $\varepsilon^2$:

$$\frac{\partial \rho_{vs}^{*(0)}}{\partial t^*} - \mathrm{div}_{y^*}(D_v^*(\mathbf{grad}_{y^*}\rho_v^{*(2)} + \mathbf{grad}_{x^*}\rho_v^{*(1)})) - \mathrm{div}_{x^*}(D_v^*(\mathbf{grad}_{y^*}\rho_v^{*(1)} + \mathbf{grad}_{x^*}\rho_{vs}^{*(0)}(T^{*(0)}))) = 0 \quad \text{in } \Omega_i \tag{B1.42}$$

$$D_v^*(\mathbf{grad}_{y^*}\rho_v^{*(2)} + \mathbf{grad}_{x^*}\rho_v^{*(1)}) \cdot \mathbf{n_i} = \rho_i^* w_n^{*(1)} \quad \text{on } \Gamma \tag{B1.43}$$

where the unknowns $\rho_v^{*(2)}(\mathbf{x}^*,\mathbf{y}^*,t)$ is $\mathbf{y}^*$-periodic and $w_n^{*(1)}$ is the normal interface velocity due to the sublimation/deposition process at the first order. Consequently, integrating (B1.42) over $\Omega_a$, and then using the divergence theorem, the periodicity condition, and the boundary conditions (B1.43) leads to the first order dimensionless description:

$$\phi\frac{\partial\rho_{vs}^{*(0)}}{\partial t}-\mathrm{div}_{x^*}(\mathbf{D}^{\mathrm{eff}*}\mathbf{grad}_{x^*}\rho_{vs}^{*(0)}(T^{*(0)}))=\int_\Gamma\rho_i^* w_n^{*(1)}dS=\rho_i^*\dot{\phi} \tag{B1.44}$$

where $\mathbf{D}^{\mathrm{eff}*}$ is the classical dimensionless effective diffusion tensor defined as (see Case A):

$$\mathbf{D}^{\mathrm{eff}*}=\frac{1}{|\Omega|}\int_{\Omega_a}D_v^*(\mathbf{grad}_{y^*}\ \mathbf{g}_v^*(\mathbf{y}^*)+\mathbf{I})\mathrm{d}\,\Omega \tag{B1.45}$$

### S.2.2 Case B2

Taking into account of the order of magnitude of the dimensionless numbers, $\left[\mathrm{F}_i^T\right]=\mathcal{O}\left(\left[\mathrm{F}_a^T\right]\right)=\mathcal{O}\left(\left[\mathrm{F}_a^\rho\right]\right)=\mathcal{O}(\varepsilon^2)$, $[\mathrm{K}]=\mathcal{O}(1)\,,[\mathrm{H}]=\mathcal{O}(1)\,,[\mathrm{W_R}]=\mathcal{O}(1)$, the dimensionless microscopic description (13)-(18) becomes:

$$\varepsilon^2\rho_i^*C_i^*\frac{\partial T_i^*}{\partial t^*}-\mathrm{div}^*(k_i^*\mathbf{grad}^*T_i^*)=0\quad\text{in }\Omega_i \tag{B2.1}$$

$$\varepsilon^2\rho_a^*C_a^*\frac{\partial T_a^*}{\partial t^*}-\mathrm{div}^*(k_a^*\mathbf{grad}T_a^*)=0\quad\text{in }\Omega_a \tag{B2.2}$$

$$\varepsilon^2\frac{\partial\rho_v^*}{\partial t^*}-\mathrm{div}^*(D_v^*\mathbf{grad}^*\rho_v^*)=0\quad\text{in }\Omega_a \tag{B2.3}$$

$$T_i^*=T_a^*\quad\text{on }\Gamma \tag{B2.4}$$

$$k_i^*\mathbf{grad}^*T_i^*\cdot\mathbf{n_i}-k_a^*\mathbf{grad}^*T_a^*\cdot\mathbf{n_i}=L_{sg}^*\frac{D_v^*}{\rho_i^*}\mathbf{grad}^*\rho_v^*\cdot\mathbf{n_i}\quad\text{on }\Gamma \tag{B2.5}$$

$$D_v^*\mathbf{grad}^*\rho_v^*\cdot\mathbf{n_i}=\rho_i^*\mathbf{w}^*\cdot\mathbf{n_i}\quad\text{on }\Gamma. \tag{B2.6}$$

This set of equations is completed by the Hertz-Knudsen equation (A.7) and the Clausius Clapeyron's law (9) expressed in dimensionless form as:

$$w_n^*=\mathbf{w}^*\cdot\mathbf{n_i}=\frac{\alpha^*}{\rho_i^*}w_\mathrm{k}^*\left[\rho_v^*-\rho_{vs}^*(T_a^*)\right]\ \text{ on }\Gamma \tag{B2.7}$$

$$\rho_{vs}^*(T_a^*)=\rho_{vs}^{\mathrm{ref}*}(T^{\mathrm{ref}*})\exp\left[\frac{L_{sg}^*m^*}{\rho_i^*k^*}\left(\frac{1}{T^{\mathrm{ref}*}}-\frac{1}{T_a^*}\right)\right] \tag{B2.8}$$

### S.2.2.1 Heat and water vapor transfer at the first order

Introducing asymptotic expansions for $T_i^*$ and $T_a^*$ in the relations (B2.1), (B2.2), (B2.4), and (B2.5) gives at the lowest order:

$$\mathrm{div}_{y^*}(k_i^* \mathbf{grad}_{y^*} T_i^{*(0)}) = 0 \quad \text{in } \Omega_i \tag{B2.9}$$

$$\mathrm{div}_{y^*}(k_a^* \mathbf{grad}_{y^*} T_a^{*(0)}) = 0 \quad \text{in } \Omega_a \tag{B2.10}$$

$$T_i^{*(0)} = T_a^{*(0)} \quad \text{on } \Gamma \tag{B2.11}$$

$$(k_i^* \mathbf{grad}_{y^*}^* T_i^{*(0)} - k_a^* \mathbf{grad}_{y^*}^* T_a^{*(0)}) \cdot \mathbf{n_i} = \frac{L_{sg}^*}{\rho_i^*} D_v^* \mathbf{grad}_{y^*} \rho_v^{*(0)} \cdot \mathbf{n_i} \quad \text{on } \Gamma \tag{B2.12}$$

where the unknowns $T_i^{*(0)}(\mathbf{x}^*, \mathbf{y}^*, t)$ and $T_a^{*(0)}(\mathbf{x}^*, \mathbf{y}^*, t)$ are $\mathbf{y}^*$-periodic. Introducing asymptotic expansions for $\rho_v^*$ in the relations (B2.3) and (B2.6) gives at the lowest order:

$$\mathrm{div}_{y^*}(D_v^* \mathbf{grad}_{y^*} \rho_v^{*(0)}) = 0 \quad \text{in } \Omega_a \tag{B2.13}$$

$$D_v^* \mathbf{grad}_{y^*} \rho_v^{*(0)} \cdot \mathbf{n_i} = \alpha^* w_{\mathrm{k}}^* \left[ \rho_v^{*(0)} - \rho_{vs}^{*(0)}(T^{*(0)}) \right] \quad \text{on } \Gamma. \tag{B2.14}$$

where the unknown $\rho_v^{*(0)}(\mathbf{x}^*, \mathbf{y}^*, t)$ is $\mathbf{y}^*$-periodic. Consequently, integrating (B2.13) over $\Omega_a$, and then using the divergence theorem, the periodicity condition, the boundary conditions (B2.14) leads to:

$$\rho_v^{*(0)} = \rho_{vs}^{*(0)} \quad \text{on } \Gamma. \tag{B2.15}$$

Taking into account this result, the solution of the above boundary value problem is given by:

$$T_i^{*(0)} = T_a^{*(0)} = T^{*(0)}(\mathbf{x}^*, t). \tag{B2.16}$$

and

$$\rho_v^{*(0)} = \rho_v^{*(0)}(\mathbf{x}^*, t) = \rho_{vs}^{*(0)}(T^{*(0)}). \tag{B2.17}$$

At the first order, the temperature and the the water vapor density are independent of the microscopic dimensionless variable $\mathbf{y}^*$, i.e. we have only one temperature field.

#### S.2.2.2 Heat and water vapor transfer at the second order

Taking into account of these results, Eq. (B2.1), (B2.2), (B2.4), and (B2.5) of order $\varepsilon$ give the following second-order problem:

$$\mathrm{div}_{y^*}(k_i^*(\mathbf{grad}_{y^*}T_i^{*(1)}+\mathbf{grad}_{x^*}T^{*(0)}))=0 \quad \text{in } \Omega_i \tag{B2.18}$$

$$\mathrm{div}_{y^*}(k_a^*(\mathbf{grad}_{y^*}T_a^{*(1)}+\mathbf{grad}_{x^*}T^{*(0)}))=0 \quad \text{in } \Omega_a \tag{B2.19}$$

$$T_i^{*(1)}=T_a^{*(1)} \quad \text{on } \Gamma \tag{B2.20}$$

$$(k_i^*(\mathbf{grad}_{y^*}T_i^{*(1)}+\mathbf{grad}_{x^*}T^{*(0)})-k_a^*(\mathbf{grad}_{y^*}T_a^{*(1)}+\mathbf{grad}_{x^*}T^{*(0)}))\cdot\mathbf{n_i}=\frac{L_{sg}^*}{\rho_i^*}D_v^*(\mathbf{grad}_{y^*}\rho_v^{*(1)}+\mathbf{grad}_{x^*}\rho_v^{*(0)})\cdot\mathbf{n_i} \quad \text{on } \Gamma \tag{B2.21}$$

where the unknowns $T_i^{*(1)}(\mathbf{x}^*,\mathbf{y}^*,t)$ and $T_a^{*(1)}(\mathbf{x}^*,\mathbf{y}^*,t)$ are $\mathbf{y}^*$-periodic and the macroscopic gradient $\mathbf{grad}_{x^*}T^{*(0)}$ is given. The second-order problem for the water vapor is given by Eq. (B2.3) and (B2.6) of order $\varepsilon$:

$$\mathrm{div}_{y^*}(D_v^*(\mathbf{grad}_{y^*}\rho_v^{*(1)}+\mathbf{grad}_{x^*}\rho_v^{*(0)}))=0 \quad \text{in } \Omega_a \tag{B2.22}$$

$$D_v^*(\mathbf{grad}_{y^*}\rho_v^{*(1)}+\mathbf{grad}_{x^*}\rho_v^{*(0)})\cdot\mathbf{n_i}=\alpha^* w_{\mathrm{k}}^*\left[\rho_v^{*(1)}-\rho_{vs}^{*(1)}\right] \quad \text{on } \Gamma. \tag{B2.23}$$

where the unknowns $\rho_v^{*(1)}(\mathbf{x}^*,\mathbf{y}^*,t)$ is $\mathbf{y}^*$-periodic. Consequently, integrating (B2.22) over $\Omega_a$, and then using the divergence theorem, the periodicity condition, the boundary conditions (B2.23) leads to:

$$\rho_v^{*(1)}=\rho_{vs}^{*(1)} \quad \text{on } \Gamma. \tag{B2.24}$$

This result also implies that

$$D_v^*(\mathbf{grad}_{y^*}\rho_v^{*(1)}+\mathbf{grad}_{x^*}\rho_v^{*(0)})\cdot\mathbf{n_i}=0 \quad \text{on } \Gamma. \tag{B2.25}$$

The above boundary value problem can not be solved by considering both boundary conditions (B2.24) and (B2.25) on the ice-air interface simultaneously. However, it seems reasonable to assume that in the range $\varepsilon^{1/2}\leqslant[\mathrm{W_R}]\leqslant 1$ the Neumann boundary condition (B2.25) can be privileged (Case B2), whereas in the range $1\leqslant[\mathrm{W_R}]\leqslant\varepsilon^{-1/2}$ the Dirichlet boundary condition (B2.24) can be considered (Case C1). These two cases will give us the two extreme behaviours when $[\mathrm{W_R}]=\mathcal{O}(1)$.

By considering first the Neumann boundary condition (B2.25), the solution of the above boundary value problem for the temperature appears as a linear function of the macroscopic gradient, modulo an arbitrary function $\widetilde{T}^{*(1)}(\mathbf{x}^*,t)$:

$$T_i^{*(1)}(\mathbf{x}^*,\mathbf{y}^*,t) = \mathbf{t}_i^*(\mathbf{y}^*)\cdot\mathbf{grad}_{x^*}T^{*(0)} + \widetilde{T}_i^{*(1)} \tag{B2.26}$$

$$T_a^{*(1)}(\mathbf{x}^*,\mathbf{y}^*,t) = \mathbf{t}_a^*(\mathbf{y}^*)\cdot\mathbf{grad}_{x^*}T^{*(0)} + \widetilde{T}_a^{*(1)} \tag{B2.27}$$

where $\mathbf{t}_i^*(\mathbf{y}^*)$ and $\mathbf{t}_a^*(\mathbf{y}^*)$ are two periodic vectors which characterize the fluctuation of temperature in both phases at the pore scale. Introducing (B2.26) and (B2.27) in the set (B2.18)-(B2.21), these two vectors are solution of the following boundary value problem in a compact form:

$$\mathrm{div}_{y^*}(k_i^*(\mathbf{grad}_{y^*}\mathbf{t}_i^* + \mathbf{I})) = 0 \quad \text{in } \Omega_i \tag{B2.28}$$

$$\mathrm{div}_{y^*}(k_a^*(\mathbf{grad}_{y^*}\mathbf{t}_a^* + \mathbf{I})) = 0 \quad \text{in } \Omega_a \tag{B2.29}$$

$$\mathbf{t}_i^* = \mathbf{t}_a^* \quad \text{on } \Gamma \tag{B2.30}$$

$$(k_i^*(\mathbf{grad}_{y^*}\mathbf{t}_i^* + \mathbf{I}) - k_a^*(\mathbf{grad}_{y^*}\mathbf{t}_a^* + \mathbf{I}))\cdot\mathbf{n_i} = 0 \quad \text{on } \Gamma \tag{B2.31}$$

$$\frac{1}{|\Omega|}\int_\Omega(\mathbf{t}_\mathrm{a}^* + \mathbf{t}_\mathrm{i}^*)\mathrm{d}\Omega = \mathbf{0} \tag{B2.32}$$

This latter equation is introduced to ensure the uniqueness of the solution. Similarly, solution of the boundary value problem (B2.22) and (B2.25) appears as a linear function of the macroscopic gradient $\mathbf{grad}_{x^*}\rho_{vs}^{*(0)}(T^{*(0)})$ modulo an arbitrary function $\widetilde{\rho}_v^{*(1)}(\mathbf{x}^*,t)$ :

$$\rho_v^{*(1)}(\mathbf{x}^*,\mathbf{y}^*,t) = \mathbf{g}_v^*(\mathbf{y}^*)\cdot\mathbf{grad}_{x^*}\rho_{vs}^{*(0)}(T^{*(0)}) + \widetilde{\rho}_v^{*(1)}(\mathbf{x}^*,t) \tag{B2.33}$$

where $\mathbf{g}_v^*(\mathbf{y}^*)$ is a periodic vector which characterizes the fluctuation of water vapor density in the air phase at the pore scale induced by the macroscopic gradient $\mathbf{grad}_{x^*}\rho_{vs}^{*(0)}(T^{*(0)})$. Introducing (B2.33) in the set (B2.22)-(B2.25), this vector is solution of the following boundary value problem in a compact form:

$$\mathrm{div}_{y^*}(D_v^*(\mathbf{grad}_{y^*}\mathbf{g}_v^* + \mathbf{I})) = 0 \quad \text{in } \Omega_a \tag{B2.34}$$

$$D_v^*(\mathbf{grad}_{y^*}\mathbf{g}_v^* + \mathbf{I}) \cdot \mathbf{n_i} = 0 \quad \text{on } \Gamma \tag{B2.35}$$

$$\frac{1}{|\Omega|}\int\limits_{\Omega_a} \mathbf{g}_v^* \mathrm{d}\Omega = \mathbf{0} \tag{B2.36}$$

This latter equation is introduced to ensure the uniqueness of the solution.

### S.2.2.3 Macroscopic description

Finally, the third order problem for the heat transfer is given by Eq. (B2.1), (B2.2), (B2.4), and (B2.5) of order $\varepsilon^2$:

$$\rho_i^* C_i^* \frac{\partial T^{*(0)}}{\partial t^*} - \mathrm{div}_{y^*}(k_i^*(\mathbf{grad}_{y^*} T_i^{*(2)} + \mathbf{grad}_{x^*} T_i^{*(1)})) - \mathrm{div}_{x^*}(k_i^*(\mathbf{grad}_{y^*} T_i^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)})) = 0 \quad \text{in } \Omega_i \tag{B2.37}$$

$$\rho_a^* C_a^* \frac{\partial T^{*(0)}}{\partial t^*} - \mathrm{div}_{y^*}(k_a^*(\mathbf{grad}_{y^*} T_a^{*(2)} + \mathbf{grad}_{x^*} T_a^{*(1)})) - \mathrm{div}_{x^*}(k_a^*(\mathbf{grad}_{y^*} T_a^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)})) = 0 \quad \text{in } \Omega_a \tag{B2.38}$$

$$T_i^{*(2)} = T_a^{*(2)} \quad \text{on}\Gamma \tag{B2.39}$$

$$\begin{aligned}(k_i^*(\mathbf{grad}_{y^*} T_i^{*(2)} + \mathbf{grad}_{x^*} T_i^{*(1)}) - k_a^*(\mathbf{grad}_{y^*} T_a^{*(2)} + \mathbf{grad}_{x^*} T_a^{*(1)})) \cdot \mathbf{n_i} \\ = L_{sg}^* \frac{D_v^*}{\rho_i^*}(\mathbf{grad}_{y^*} \rho_v^{*(2)} + \mathbf{grad}_{x^*} \rho_v^{*(1)})) \cdot \mathbf{n_i} \quad \text{on } \Gamma\end{aligned} \tag{B2.40}$$

where the unknowns $T_i^{*(2)}(\mathbf{x}^*, \mathbf{y}^*, t)$ and $T_a^{*(2)}(\mathbf{x}^*, \mathbf{y}^*, t)$ are $\mathbf{y}^*$-periodic. Integrating (B2.37) over $\Omega_i$ and (B2.38) over $\Omega_a$, and then using the divergence theorem, the periodicity condition, and the boundary conditions (B2.40) and the results leads to the first order dimensionless description:

$$(\rho C)^{\mathrm{eff}*} \frac{\partial T^{*(0)}}{\partial t^*} - \mathrm{div}_{x^*}(\mathbf{k}^{\mathrm{eff}*} \mathbf{grad}_{x^*} T^{*(0)}) = -\int\limits_{\Gamma} L_{sg}^* w_n^{*(2)} dS = L_{sg}^* \dot{\phi} \tag{B2.41}$$

where $(\rho C)^{\mathrm{eff}*}$ and $\mathbf{k}^{\mathrm{eff}*}$ are the dimensionless effective thermal capacity and the effective dimensionless thermal conductivity respectively, defined as:

$$(\rho C)^{\mathrm{eff}*} = (1-\phi)\rho_i^* C_i^* + \phi \rho_a^* C_a^* \tag{B2.42}$$

$$\mathbf{k}^{\mathrm{eff}*} = \frac{1}{|\Omega|}\left(\int\limits_{\Omega_\mathrm{a}} k_a^*(\mathbf{grad}_{y^*}\mathbf{t}_a^*(\mathbf{y}^*) + \mathbf{I})\mathrm{d}\Omega + \int\limits_{\Omega_\mathrm{i}} k_i^*(\mathbf{grad}_{y^*}\mathbf{t}_i^*(\mathbf{y}^*) + \mathbf{I})\mathrm{d}\Omega\right) \tag{B2.43}$$

where $\phi$ is the porosity.

Finally, the third order problem for the water vapor is given by Eq. (B2.3) and (B2.6) of order $\varepsilon^2$:

$$\frac{\partial \rho_{vs}^{*(0)}}{\partial t^*} - \mathrm{div}_{y^*}(D_v^*(\mathbf{grad}_{y^*}\rho_v^{*(2)} + \mathbf{grad}_{x^*}\rho_v^{*(1)})) - \mathrm{div}_{x^*}(D_v^*(\mathbf{grad}_{y^*}\rho_v^{*(1)} + \mathbf{grad}_{x^*}\rho_{vs}^{*(0)}(T^{*(0)}))) = 0 \quad \text{in } \Omega_i \tag{B2.44}$$

$$D_v^*(\mathbf{grad}_{y^*}\rho_v^{*(2)} + \mathbf{grad}_{x^*}\rho_v^{*(1)}) \cdot \mathbf{n_i} = \rho_i^* w_n^{*(2)} \quad \text{on } \Gamma \tag{B2.45}$$

where the unknown $\rho_v^{*(2)}(\mathbf{x}^*, \mathbf{y}^*, t)$ is $\mathbf{y}^*$-periodic and $w_n^{*(2)}$ is the normal interface velocity due to the sublimation/deposition process at the first order. Consequently, integrating (B2.44) over $\Omega_a$, and then using the divergence theorem, the periodicity condition, and the boundary conditions (B2.45) leads to the first order dimensionless description:

$$\phi\frac{\partial \rho_{vs}^{*(0)}}{\partial t} - \mathrm{div}_{x^*}(\mathbf{D}^{\mathrm{eff}*}\mathbf{grad}_{x^*}\rho_{vs}^{*(0)}(T^{*(0)})) = \int_\Gamma \rho_i^* w_n^{*(2)} dS = \rho_i^* \dot{\phi} \tag{B2.46}$$

where $\mathbf{D}^{\mathrm{eff}*}$ is the classical dimensionless effective diffusion tensor defined as:

$$\mathbf{D}^{\mathrm{eff}*} = \frac{1}{|\Omega|}\int_{\Omega_a} D_v^*(\mathbf{grad}_{y^*}\ \mathbf{g}_v^*(\mathbf{y}^*) + \mathbf{I})\mathrm{d}\,\Omega \tag{B2.47}$$

## S.3 Cases C1, C2 et C3

### S.3.1 Case C1

#### S.3.1.1 Heat and water vapor transfer at the second order

The Dirichlet boundary condition (B2.24) is now considered. According to (A.40), this boundary condition can be also written as:

$$\rho_v^{*(1)} = \rho_{vs}^{*(1)} = \gamma^*(T^{*(0)})T_a^{*(1)} \quad \text{on } \Gamma \tag{C1.1}$$

Moreover, we have

$$\mathbf{grad}_{x^*}\rho_{vs}^{*(0)} = \gamma^*(T^{*(0)})\mathbf{grad}_{x^*}T^{*(0)} \tag{C1.2}$$

thus Eq. (B2.22) and (B2.24) are written:

$$\mathrm{div}_{y^*}(D_v^*(\mathbf{grad}_{y^*}\rho_v^{*(1)} + \gamma^*(T^{*(0)})\mathbf{grad}_{x^*}T^{*(0)}) = 0 \quad \text{in } \Omega_a \tag{C1.3}$$

$$\rho_v^{*(1)} = \gamma^*(T^{*(0)})T_a^{*(1)} \quad \text{on } \Gamma \tag{C1.4}$$

The solution of the boundary value problems (B2.18)-(B2.21) and (C1.3)-(C1.4) appears as a linear function of the macroscopic gradient $\mathbf{grad}_{x^*} T^{*(0)}$, modulo an arbitrary function.

$$T_i^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t) = \mathbf{r}_i^*(\mathbf{y}^*) \cdot \mathbf{grad}_{x^*} T^{*(0)} + \widetilde{T}_i^{*(1)} \tag{C1.5}$$

$$T_a^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t) = \mathbf{r}_a^*(\mathbf{y}^*) \cdot \mathbf{grad}_{x^*} T^{*(0)} + \widetilde{T}_a^{*(1)} \tag{C1.6}$$

$$\rho_v^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t) = \gamma^*(T^{*(0)}) \, (\mathbf{r}_a^*(\mathbf{y}^*) \cdot \mathbf{grad}_{x^*} T^{*(0)} + \widetilde{T}_a^{*(1)}) \quad \text{on } \Gamma \tag{C1.7}$$

where $\mathbf{r}_i^*(\mathbf{y}^*)$ and $\mathbf{r}_a^*(\mathbf{y}^*)$ are two periodic vectors which characterize the fluctuation of temperature in both phases at the pore scale. Introducing (C1.5) and (C1.6) in the set (B2.18)-(B2.21) and (C1.3)-(C1.4), these two vectors are solution of the following boundary value problem in a compact form:

$$\mathrm{div}_{y^*}(k_i^*(\mathbf{grad}_{y^*} \mathbf{r}_i^* + \mathbf{I})) = 0 \quad \text{in } \Omega_i \tag{C1.8}$$

$$\mathrm{div}_{y^*}((k_a^* + L_{sg}^* D_v^* \frac{\gamma^*(T^{*(0)})}{\rho_i^*})(\mathbf{grad}_{y^*} \mathbf{r}_a^* + \mathbf{I})) = 0 \quad \text{in } \Omega_a \tag{C1.9}$$

$$\mathbf{r}_i^* = \mathbf{r}_a^* \quad \text{on } \Gamma \tag{C1.10}$$

$$(k_i^*(\mathbf{grad}_{y^*} \mathbf{r}_i^* + \mathbf{I}) - (k_a^* + L_{sg}^* D_v^* \frac{\gamma^*(T^{*(0)})}{\rho_i^*})(\mathbf{grad}_{y^*} \mathbf{r}_a^* + \mathbf{I})) \cdot \mathbf{n_i} = 0 \quad \text{on } \Gamma \tag{C1.11}$$

$$\frac{1}{|\Omega|} \int_{\Omega} (\mathbf{r}_\mathrm{a}^* + \mathbf{r}_\mathrm{i}^*) \mathrm{d}\Omega = \mathbf{0} \tag{C1.12}$$

This latter equation is introduced to ensure the uniqueness of the solution. This latter boundary value problem is similar as the (A.20)-(A.24) where $k_a^*$ is now equal to $k_a^* + L_{sg}^* D_v^* \gamma^*(T^{*(0)})/\rho_i^*$. At the local scale, the thermal conductivity appears to be enhanced by the phase change.

#### S.3.1.2 Macroscopic description

Finally, the third order problem is given by Eq. (B2.1), (B2.2), (B2.4), and (B2.5) of order $\varepsilon^2$:

$$\rho_i^* C_i^* \frac{\partial T^{*(0)}}{\partial t^*} - \mathrm{div}_{y^*}(k_i^*(\mathbf{grad}_{y^*} T_i^{*(2)} + \mathbf{grad}_{x^*} T_i^{*(1)})) - \mathrm{div}_{x^*}(k_i^*(\mathbf{grad}_{y^*} T_i^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)})) = 0 \quad \text{in } \Omega_i \tag{C1.13}$$

$$\rho_a^* C_a^* \frac{\partial T^{*(0)}}{\partial t^*} - \mathrm{div}_{y^*}(k_a^*(\mathbf{grad}_{y^*} T_a^{*(2)} + \mathbf{grad}_{x^*} T_a^{*(1)})) - \mathrm{div}_{x^*}(k_a^*(\mathbf{grad}_{y^*} T_a^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)})) = 0 \quad \text{in } \Omega_a \tag{C1.14}$$

$$T_i^{*(2)} = T_a^{*(2)} \quad \text{on}\Gamma \tag{C1.15}$$

$$(k_i^*(\mathbf{grad}_{y^*} T_i^{*(2)} + \mathbf{grad}_{x^*} T_i^{*(1)}) - k_a^*(\mathbf{grad}_{y^*} T_a^{*(2)} + \mathbf{grad}_{x^*} T_a^{*(1)})) \cdot \mathbf{n_i} = \tag{C1.16}$$
$$= L_{sg}^* \frac{D_v^*}{\rho_i^*}(\mathbf{grad}_{y^*} \rho_v^{*(2)} + \mathbf{grad}_{x^*} \rho_v^{*(1)}) \cdot \mathbf{n_i} \quad \text{on } \Gamma.$$

where the unknowns $T_i^{*(2)}(\mathbf{x}^*, \mathbf{y}^*, t)$ and $T_a^{*(2)}(\mathbf{x}^*, \mathbf{y}^*, t)$ are $\mathbf{y}^*$-periodic. For the water vapor, the third order problem is given by Eq. (B2.3) and (B2.6) of order $\varepsilon^2$:

$$\frac{\partial \rho_{vs}^{*(0)}}{\partial t^*} - \mathrm{div}_{y^*}(D_v^*(\mathbf{grad}_{y^*} \rho_v^{*(2)} + \mathbf{grad}_{x^*} \rho_v^{*(1)})) - \mathrm{div}_{x^*}(D_v^*(\mathbf{grad}_{y^*} \rho_v^{*(1)} + \mathbf{grad}_{x^*} \rho_{vs}^{*(0)}(T^{*(0)}))) = 0 \quad \text{in } \Omega_a \tag{C1.17}$$

$$D_v^*(\mathbf{grad}_{y^*} \rho_v^{*(2)} + \mathbf{grad}_{x^*} \rho_v^{*(1)}) \cdot \mathbf{n_i} = \rho_i^* w_n^{*(2)} \quad \text{on } \Gamma \tag{C1.18}$$

Integrating (C1.13) over $\Omega_i$ and (C1.14) over $\Omega_a$, and then using the divergence theorem, the periodicity condition, and the boundary conditions (C1.16) leads to the first order dimensionless description:

$$(\rho C)^{\mathrm{eff}*} \frac{\partial T^{*(0)}}{\partial t^*} - \mathrm{div}_{x^*}(\mathbf{k}^{\mathrm{td}*} \mathbf{grad}_{x^*} T^{*(0)}) = \int_\Gamma L_{sg}^* \frac{D_v^*}{\rho_i^*}(\mathbf{grad}_{y^*} \rho_v^{*(2)} + \mathbf{grad}_{x^*} \rho_v^{*(1)}) \cdot \mathbf{n_i} dS = -L_{sg}^* \dot{\phi}. \tag{C1.19}$$

where $(\rho C)^{\mathrm{eff}*}$ and $\mathbf{k}^{\mathrm{td}*}$ are the dimensionless effective thermal capacity and the apparent dimensionless thermal conductivity, respectively, defined as:

$$(\rho C)^{\mathrm{eff}*} = (1-\phi)\rho_i^* C_i^* + \phi \rho_a^* C_a^* \tag{C1.20}$$

$$\mathbf{k}^{\mathrm{td}*} = \frac{1}{|\Omega|}\left( \int_{\Omega_\mathrm{a}} k_a^*(\mathbf{grad}_{y^*} \mathbf{r}_a^*(\mathbf{y}^*) + \mathbf{I}) \mathrm{d}\Omega + \int_{\Omega_\mathrm{i}} k_i^*(\mathbf{grad}_{y^*} \mathbf{r}_i^*(\mathbf{y}^*) + \mathbf{I}) \mathrm{d}\Omega \right) \tag{C1.21}$$

where $\phi$ is the porosity. Integrating (C1.17) over $\Omega_a$, and then using the divergence theorem and the periodicity condition, leads to the first order dimensionless description:

$$\phi \frac{\partial \rho_{vs}^{*(0)}}{\partial t} - \mathrm{div}_{x^*}(\mathbf{D}^{\mathrm{td}*} \mathbf{grad}_{x^*} \rho_{vs}^{*(0)}(T^{*(0)})) = -\int_\Gamma D_v^*(\mathbf{grad}_{y^*} \rho_v^{*(2)} + \mathbf{grad}_{x^*} \rho_v^{*(1)}) \cdot \mathbf{n_i} dS = \rho_i^* \dot{\phi}. \tag{C1.22}$$

where $\mathbf{D}^{\mathrm{td}*}$ is the macroscopic effective diffusion tensor defined as:

$$\mathbf{D}^{\mathrm{td}*} = \frac{1}{|\Omega|} \int_{\Omega_a} D_v^*(\mathbf{grad}_{y^*}\ \mathbf{r}_a^*(\mathbf{y}^*) + \mathbf{I}) \mathrm{d}\,\Omega \tag{C1.23}$$

### S.3.2 Case C2

Taking into account of the order of magnitude of the dimensionless numbers, $\left[\mathrm{F}_i^T\right]=\mathcal{O}\left(\left[\mathrm{F}_a^T\right]\right)=\mathcal{O}\left(\left[\mathrm{F}_a^\rho\right]\right)=\mathcal{O}(\varepsilon^2)$, $[\mathrm{K}]=\mathcal{O}\left(1\right),[\mathrm{W_R}]=\mathcal{O}\left(\varepsilon^{-1}\right),[\mathrm{H}]=\mathcal{O}\left(1\right)$, the dimensionless microscopic description (13)-(18) becomes:

$$\varepsilon^2\rho_i^*C_i^*\frac{\partial T_i^*}{\partial t^*}-\mathrm{div}^*(k_i^*\mathbf{grad}^*T_i^*)=0 \quad \text{in } \Omega_i \tag{C2.1}$$

$$\varepsilon^2\rho_a^*C_a^*\frac{\partial T_a^*}{\partial t^*}-\mathrm{div}^*(k_a^*\mathbf{grad}T_a^*)=0 \quad \text{in } \Omega_a \tag{C2.2}$$

$$\varepsilon^2\frac{\partial \rho_v^*}{\partial t^*}-\mathrm{div}^*(D_v^*\mathbf{grad}^*\rho_v^*)=0 \quad \text{in } \Omega_a \tag{C2.3}$$

$$T_i^*=T_a^* \quad \text{on } \Gamma \tag{C2.4}$$

$$k_i^*\mathbf{grad}^*T_i^*\cdot\mathbf{n_i}-k_a^*\mathbf{grad}^*T_a^*\cdot\mathbf{n_i}=L_{sg}^*\frac{D_v^*}{\rho_i^*}\mathbf{grad}^*\rho_v^*\cdot\mathbf{n_i} \quad \text{on } \Gamma \tag{C2.5}$$

$$D_v^*\mathbf{grad}^*\rho_v^*\cdot\mathbf{n_i}=\varepsilon^{-1}\rho_i^*\mathbf{w}^*\cdot\mathbf{n_i} \quad \text{on } \Gamma. \tag{C2.6}$$

This set of equations is completed by the Hertz-Knudsen equation (A.7) and the Clausius Clapeyron's law (9) expressed in dimensionless form as:

$$w_n^*=\mathbf{w}^*\cdot\mathbf{n_i}=\frac{\alpha^*}{\rho_i^*}w_\mathrm{k}^*\left[\rho_v^*-\rho_{vs}^*(T_a^*)\right] \quad \text{on } \Gamma \tag{C2.7}$$

$$\rho_{vs}^*(T_a^*)=\rho_{vs}^{\mathrm{ref}*}(T^{\mathrm{ref}*})\exp\left[\frac{L_{sg}^*m^*}{\rho_i^*k^*}\left(\frac{1}{T^{\mathrm{ref}*}}-\frac{1}{T_a^*}\right)\right] \tag{C2.8}$$

#### S.3.2.1 Heat transfer and water vapor transfer at the first and the second order

Introducing asymptotic expansions for $T_i^*$ and $T_a^*$ in the relations (C2.1), (C2.2), (C2.4), and (C2.5) give at the lowest order:

$$\mathrm{div}_{y^*}(k_i^*\mathbf{grad}_{y^*}T_i^{*(0)})=0 \quad \text{in } \Omega_i \tag{C2.9}$$

$$\mathrm{div}_{y^*}(k_a^*\mathbf{grad}_{y^*}T_a^{*(0)})=0 \quad \text{in } \Omega_a \tag{C2.10}$$

$$T_i^{*(0)} = T_a^{*(0)} \quad \text{on } \Gamma \tag{C2.11}$$

$$(k_i^* \mathbf{grad}_{y^*}^* T_i^{*(0)} - k_a^* \mathbf{grad}_{y^*}^* T_a^{*(0)}) \cdot \mathbf{n_i} = L_{sg}^* \frac{D_v^*}{\rho_i^*} \mathbf{grad}_{y^*} \rho_v^{*(0)} \quad \text{on } \Gamma \tag{C2.12}$$

where the unknowns $T_i^{*(0)}(\mathbf{x}^*, \mathbf{y}^*, t)$ and $T_a^{*(0)}(\mathbf{x}^*, \mathbf{y}^*, t)$ are $\mathbf{y}^*$-periodic. Introducing asymptotic expansions for $\rho_v^*$ in the relations (C2.3, C2.6) give at the lowest order:

$$\mathrm{div}_{y^*}(D_v^* \mathbf{grad}_{y^*} \rho_v^{*(0)}) = 0 \quad \text{in } \Omega_a \tag{C2.13}$$

$$\rho_v^{*(0)} = \rho_{vs}^{*(0)}(T^{*(0)}) \quad \text{on } \Gamma. \tag{C2.14}$$

where the unknown $\rho_v^{*(0)}(\mathbf{x}^*, \mathbf{y}^*, t)$ is $\mathbf{y}^*$-periodic. The solution of the above boundary value problems is given by:

$$\rho_v^{*(0)} = \rho_v^{*(0)}(\mathbf{x}^*, t) = \rho_{vs}^{*(0)}(T^{*(0)}). \tag{C2.15}$$

$$T_i^{*(0)} = T_a^{*(0)} = T^{*(0)}(\mathbf{x}^*, t). \tag{C2.16}$$

At the first order, the temperature and the water vapor density are independent of the microscopic dimensionless variable $\mathbf{y}^*$. We have only one temperature field. Taking into account of these results, Eq. (C2.1), (C2.2), (C2.4), and (C2.5) of order $\varepsilon$ give the following second-order problem:

$$\mathrm{div}_{y^*}(k_i^*(\mathbf{grad}_{y^*} T_i^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)})) = 0 \quad \text{in } \Omega_i \tag{C2.17}$$

$$\mathrm{div}_{y^*}(k_a^*(\mathbf{grad}_{y^*} T_a^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)})) = 0 \quad \text{in } \Omega_a \tag{C2.18}$$

$$T_i^{*(1)} = T_a^{*(1)} \quad \text{on } \Gamma \tag{C2.19}$$

$$\begin{aligned} &(k_i^*(\mathbf{grad}_{y^*} T_i^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)}) - k_a^*(\mathbf{grad}_{y^*} T_a^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)})) \cdot \mathbf{n_i} \\ &= L_{sg}^* \frac{D_v^*}{\rho_i^*}(\mathbf{grad}_{y^*} \rho_v^{*(1)} + \mathbf{grad}_{x^*} \rho_v^{*(0)}) \cdot \mathbf{n_i} \quad \text{on } \Gamma. \end{aligned} \tag{C2.20}$$

where the unknowns $T_i^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t)$ and $T_a^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t)$ are $\mathbf{y}^*$-periodic and the macroscopic gradient $\mathbf{grad}_{x^*} T^{*(0)}$ is given. Moreover we have the second-order problem for Eq. (C2.3) and (C2.6) is written:

$$\mathrm{div}_{y^*}(D_v^*(\mathbf{grad}_{y^*}\rho_v^{*(1)} + \mathbf{grad}_{x^*}\rho_{vs}^{*(0)})) = 0 \quad \text{in } \Omega_a \tag{C2.21}$$

$$\rho_v^{*(1)} = \rho_{vs}^{*(1)} \quad \text{on } \Gamma. \tag{C2.22}$$

where the unknowns $\rho_v^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t)$ is $\mathbf{y}^*$-periodic. According to (A.40), this latter boundary condition can be also written

$$\rho_v^{*(1)} = \rho_{vs}^{*(1)} = \gamma^*(T^{*(0)})T_a^{*(1)} \quad \text{on } \Gamma \tag{C2.23}$$

Moreover, we have

$$\mathbf{grad}_{x^*}\rho_{vs}^{*(0)} = \gamma^*(T^{*(0)})\mathbf{grad}_{x^*}T^{*(0)} \tag{C2.24}$$

thus Eq. (C2.21) and (C2.23) are written:

$$\mathrm{div}_{y^*}(D_v^*(\mathbf{grad}_{y^*}\rho_v^{*(1)} + \gamma^*(T^{*(0)})\mathbf{grad}_{x^*}T^{*(0)}) = 0 \quad \text{in } \Omega_a \tag{C2.25}$$

$$\rho_v^{*(1)} = \gamma^*(T^{*(0)})T_a^{*(1)} \quad \text{on } \Gamma \tag{C2.26}$$

As in the Case C1, the solution of the above boundary value problems (C2.17)-(C2.20) and (C2.25)-(C2.26) appears as a linear function of the macroscopic gradient $\mathbf{grad}_{x^*} T^{*(0)}$, modulo an arbitrary function.

$$T_i^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t) = \mathbf{r}_i^*(\mathbf{y}^*) \cdot \mathbf{grad}_{x^*}T^{*(0)} + \widetilde{T}_i^{*(1)} \tag{C2.27}$$

$$T_a^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t) = \mathbf{r}_a^*(\mathbf{y}^*) \cdot \mathbf{grad}_{x^*}T^{*(0)} + \widetilde{T}_a^{*(1)} \tag{C2.28}$$

$$\rho_v^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t) = \gamma^*(T^{*(0)})(\mathbf{r}_a^*(\mathbf{y}^*) \cdot \mathbf{grad}_{x^*}T^{*(0)} + \widetilde{T}_a^{*(1)}) \quad \text{on } \Gamma \tag{C2.29}$$

where $\mathbf{r}_i^*(\mathbf{y}^*)$ and $\mathbf{r}_a^*(\mathbf{y}^*)$ are two periodic vectors which characterize the fluctuation of temperature in both phases at the pore scale. Introducing (C2.27) and (C2.28) in the set (C2.17)-(C2.20), these two vectors are solution of the following boundary value problem in a compact form:

$$\mathrm{div}_{y^*}(k_i^*(\mathbf{grad}_{y^*}\mathbf{r}_i^* + \mathbf{I})) = 0 \quad \text{in } \Omega_i \tag{C2.30}$$

$$\mathrm{div}_{y^*}(k_a^*(\mathbf{grad}_{y^*}\mathbf{r}_a^* + \mathbf{I})) = 0 \quad \text{in } \Omega_a \tag{C2.31}$$

$$\mathbf{r}_i^* = \mathbf{r}_a^* \quad \text{on } \Gamma \tag{C2.32}$$

$$(k_i^*(\mathbf{grad}_{y^*}\mathbf{r}_i^* + \mathbf{I}) - (k_a^* + L_{sg}^* D_v^* \frac{\gamma^*(T^{*(0)})}{\rho_i^*})(\mathbf{grad}_{y^*}\mathbf{r}_a^* + \mathbf{I})) \cdot \mathbf{n_i} = 0 \quad \text{on } \Gamma \tag{C2.33}$$

$$\frac{1}{|\Omega|}\int_{\Omega}(\mathbf{r}_{\mathrm{a}}^* + \mathbf{r}_{\mathrm{i}}^*)\mathrm{d}\Omega = \mathbf{0} \tag{C2.34}$$

This latter equation is introduced to ensure the uniqueness of the solution. This latter boundary value problem is similar as the (A.20)-(A.24) where $k_a^*$ is now equal to $k_a^* + L_{sg}^* D_v^* \gamma^*(T^{*(0)})/\rho_i^*$. At the local scale, the thermal conductivity appears to be enhanced by the phase change.

### S.3.2.2 Macroscopic description

Finally, the third order problem is given by the equations (C2.1, C2.2, C2.4, C2.5) of order $\varepsilon^2$:

$$\rho_i^* C_i^* \frac{\partial T^{*(0)}}{\partial t^*} - \mathrm{div}_{y^*}(k_i^*(\mathbf{grad}_{y^*}T_i^{*(2)} + \mathbf{grad}_{x^*}T_i^{*(1)})) - \mathrm{div}_{x^*}(k_i^*(\mathbf{grad}_{y^*}T_i^{*(1)} + \mathbf{grad}_{x^*}T^{*(0)})) = 0 \quad \text{in } \Omega_i \tag{C2.35}$$

$$\rho_a^* C_a^* \frac{\partial T^{*(0)}}{\partial t^*} - \mathrm{div}_{y^*}(k_a^*(\mathbf{grad}_{y^*}T_a^{*(2)} + \mathbf{grad}_{x^*}T_a^{*(1)})) - \mathrm{div}_{x^*}(k_a^*(\mathbf{grad}_{y^*}T_a^{*(1)} + \mathbf{grad}_{x^*}T^{*(0)})) = 0 \quad \text{in } \Omega_a \tag{C2.36}$$

$$T_i^{*(2)} = T_a^{*(2)} \quad \text{on}\Gamma \tag{C2.37}$$

$$(k_i^*(\mathbf{grad}_{y^*}T_i^{*(2)} + \mathbf{grad}_{x^*}T_i^{*(1)}) - k_a^*(\mathbf{grad}_{y^*}T_a^{*(2)} + \mathbf{grad}_{x^*}T_a^{*(1)})) \cdot \mathbf{n_i} = \tag{C2.38}$$
$$= L_{sg}^* \frac{D_v^*}{\rho_i^*}(\mathbf{grad}_{y^*}\rho_v^{*(2)} + \mathbf{grad}_{x^*}\rho_v^{*(1)}) \cdot \mathbf{n_i} \quad \text{on } \Gamma.$$

where the unknowns $T_i^{*(2)}(\mathbf{x}^*, \mathbf{y}^*, t)$ and $T_a^{*(2)}(\mathbf{x}^*, \mathbf{y}^*, t)$ are $\mathbf{y}^*$-periodic. For the water vapor, the third order problem is given by the the equations (C2.3, C2.6) of order $\varepsilon^2$:

$$\frac{\partial \rho_{vs}^{*(0)}}{\partial t^*} - \mathrm{div}_{y^*}(D_v^*(\mathbf{grad}_{y^*}\rho_v^{*(2)} + \mathbf{grad}_{x^*}\rho_v^{*(1)})) - \mathrm{div}_{x^*}(D_v^*(\mathbf{grad}_{y^*}\rho_v^{*(1)} + \mathbf{grad}_{x^*}\rho_{vs}^{*(0)}(T^{*(0)}))) = 0 \quad \text{in } \Omega_a \tag{C2.39}$$

$$D_v^*(\mathbf{grad}_{y^*}\rho_v^{*(2)} + \mathbf{grad}_{x^*}\rho_v^{*(1)}) \cdot \mathbf{n_i} = \rho_i^* w_n^{*(3)} \quad \text{on } \Gamma \tag{C2.40}$$

Integrating (C2.35) over $\Omega_i$ and (C2.36) and (C2.39) over $\Omega_a$, and then using the divergence theorem, the periodicity condition, and the boundary conditions (B2.40) leads to the first order dimensionless description:

$$(\rho C)^{\text{eff}*}\frac{\partial T^{*(0)}}{\partial t^*} - \text{div}_{x^*}(\mathbf{k}^{\text{td}*}\mathbf{grad}_{x^*}T^{*(0)}) = \int_\Gamma L_{sg}^*\frac{D_v^*}{\rho_i^*}(\mathbf{grad}_{y^*}\rho_v^{*(2)} + \mathbf{grad}_{x^*}\rho_v^{*(1)}) \cdot \mathbf{n_i} dS = -L_{sg}^*\dot{\phi}. \tag{C2.41}$$

where $(\rho C)^{\text{eff}*}$ and $\mathbf{k}^{\text{td}*}$ are the dimensionless effective thermal capacity and the apparent dimensionless conductivity respectively, defined as:

$$(\rho C)^{\text{eff}*} = (1-\phi)\rho_i^* C_i^* + \phi\rho_a^* C_a^* \tag{C2.42}$$

$$\mathbf{k}^{\text{td}*} = \frac{1}{|\Omega|}\left(\int_{\Omega_\text{a}} k_a^*(\mathbf{grad}_{y^*}\mathbf{r}_a^*(\mathbf{y}^*) + \mathbf{I})\text{d}\Omega + \int_{\Omega_\text{i}} k_i^*(\mathbf{grad}_{y^*}\mathbf{r}_i^*(\mathbf{y}^*) + \mathbf{I})\text{d}\Omega\right) \tag{C2.43}$$

where $\phi$ is the porosity. Integrating (C2.39) over $\Omega_a$, and then using the divergence theorem and the periodicity condition, leads to the first order dimensionless description:

$$\phi\frac{\partial\rho_{vs}^{*(0)}}{\partial t} - \text{div}_{x^*}(\mathbf{D}^{\text{td}*}\mathbf{grad}_{x^*}\rho_{vs}^{*(0)}(T^{*(0)})) = -\int_\Gamma D_v^*(\mathbf{grad}_{y^*}\rho_v^{*(2)} + \mathbf{grad}_{x^*}\rho_v^{*(1)}) \cdot \mathbf{n_i} dS = \rho_i^*\dot{\phi}. \tag{C2.44}$$

where $\mathbf{D}^{\text{td}*}$ is the apparent effective diffusion tensor defined as:

$$\mathbf{D}^{\text{td}*} = \frac{1}{|\Omega|}\int_{\Omega_a} D_v^*(\mathbf{grad}_{y^*}\ \mathbf{r}_a^*(\mathbf{y}^*) + \mathbf{I})\text{d}\ \Omega \tag{C2.45}$$

### S.3.3 Case C3

Taking into account of the order of magnitude of the dimensionless numbers, $\left[\text{F}_i^T\right] = \mathcal{O}\left(\left[\text{F}_a^T\right]\right) = \mathcal{O}\left(\left[\text{F}_a^\rho\right]\right) = \mathcal{O}(\varepsilon^2)$, $[\text{K}] = \mathcal{O}(1)$, $[\text{W}_\text{R}] = \mathcal{O}\left(\varepsilon^{-1}\right)$, $[\text{H}] = \mathcal{O}(1)$, the dimensionless microscopic description (13)-(18) becomes:

$$\varepsilon^2\rho_i^* C_i^*\frac{\partial T_i^*}{\partial t^*} - \text{div}^*(k_i^*\mathbf{grad}^* T_i^*) = 0 \quad \text{in } \Omega_i \tag{C3.1}$$

$$\varepsilon^2\rho_a^* C_a^*\frac{\partial T_a^*}{\partial t^*} - \text{div}^*(k_a^*\mathbf{grad} T_a^*) = 0 \quad \text{in } \Omega_a \tag{C3.2}$$

$$\varepsilon^2\frac{\partial\rho_v^*}{\partial t^*} - \text{div}^*(D_v^*\mathbf{grad}^*\rho_v^*) = 0 \quad \text{in } \Omega_a \tag{C3.3}$$

$$T_i^* = T_a^* \quad \text{on } \Gamma \tag{C3.4}$$

$$k_i^* \mathbf{grad}^* T_i^* \cdot \mathbf{n_i} - k_a^* \mathbf{grad}^* T_a^* \cdot \mathbf{n_i} = L_{sg}^* \frac{D_v^*}{\rho_i^*} \mathbf{grad}^* \rho_v^* \cdot \mathbf{n_i} \quad \text{on } \Gamma \tag{C3.5}$$

$$D_v^* \mathbf{grad}^* \rho_v^* \cdot \mathbf{n_i} = \varepsilon^{-2} \rho_i^* \mathbf{w}^* \cdot \mathbf{n_i} \quad \text{on } \Gamma. \tag{C3.6}$$

This set of equations is completed by the Hertz-Knudsen equation (A.7) and the Clausius Clapeyron's law (9) expressed in dimensionless form as:

$$w_n^* = \mathbf{w}^* \cdot \mathbf{n_i} = \frac{\alpha^*}{\rho_i^*} w_\mathrm{k}^* \left[\rho_v^* - \rho_{vs}^*(T_a^*)\right] \;\; \text{on } \Gamma \tag{C3.7}$$

$$\rho_{vs}^*(T_a^*) = \rho_{vs}^{\mathrm{ref}*}(T^{\mathrm{ref}*}) \exp\left[\frac{L_{sg}^* m^*}{\rho_i^* k^*}\left(\frac{1}{T^{\mathrm{ref}*}} - \frac{1}{T_a^*}\right)\right] \tag{C3.8}$$

#### S.3.3.1 Heat transfer and water vapor transfer at the first and second order

Introducing asymptotic expansions for $T_i^*$ and $T_a^*$ in the relations (C3.1, C3.2, C3.4, C3.5) give at the lowest order:

$$\mathrm{div}_{y^*}(k_i^* \mathbf{grad}_{y^*} T_i^{*(0)}) = 0 \quad \text{in } \Omega_i \tag{C3.9}$$

$$\mathrm{div}_{y^*}(k_a^* \mathbf{grad}_{y^*} T_a^{*(0)}) = 0 \quad \text{in } \Omega_a \tag{C3.10}$$

$$T_i^{*(0)} = T_a^{*(0)} \quad \text{on } \Gamma \tag{C3.11}$$

$$(k_i^* \mathbf{grad}_{y^*}^* T_i^{*(0)} - k_a^* \mathbf{grad}_{y^*}^* T_a^{*(0)}) \cdot \mathbf{n_i} = L_{sg}^* \frac{D_v^*}{\rho_i^*} \mathbf{grad}_{y^*} \rho_v^{*(0)} \quad \text{on } \Gamma \tag{C3.12}$$

where the unknowns $T_i^{*(0)}(\mathbf{x}^*, \mathbf{y}^*, t)$ and $T_a^{*(0)}(\mathbf{x}^*, \mathbf{y}^*, t)$ are $\mathbf{y}^*$-periodic. Introducing asymptotic expansions for $\rho_v^*$ in the relations (C3.3, C3.6) give at the lowest order

$$\mathrm{div}_{y^*}(D_v^* \mathbf{grad}_{y^*} \rho_v^{*(0)}) = 0 \quad \text{in } \Omega_a \tag{C3.13}$$

$$\rho_v^{*(0)} = \rho_{vs}^{*(0)}(T^{*(0)}) \quad \text{on } \Gamma. \tag{C3.14}$$

where the unknowns $\rho_v^{*(0)}(\mathbf{x}^*, \mathbf{y}^*, t)$ is $\mathbf{y}^*$-periodic. The solution of the above boundary value problems is given by:

$$\rho_v^{*(0)} = \rho_v^{*(0)}(\mathbf{x}^*, t) = \rho_{vs}^{*(0)}(T^{*(0)}). \tag{C3.15}$$

$$T_i^{*(0)} = T_a^{*(0)} = T^{*(0)}(\mathbf{x}^*, t). \tag{C3.16}$$

At the first order, the temperature and the water vapor density are independent of the microscopic dimensionless variable $\mathbf{y}^*$. We have only one temperature field. Taking into account of these results, equations (C3.1, C3.2, C3.4, C3.5) of order $\varepsilon$ give the following second-order problem:

$$\mathrm{div}_{y^*}(k_i^*(\mathbf{grad}_{y^*} T_i^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)})) = 0 \quad \text{in } \Omega_i \tag{C3.17}$$

$$\mathrm{div}_{y^*}(k_a^*(\mathbf{grad}_{y^*} T_a^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)})) = 0 \quad \text{in } \Omega_a \tag{C3.18}$$

$$T_i^{*(1)} = T_a^{*(1)} \quad \text{on } \Gamma \tag{C3.19}$$

$$(k_i^*(\mathbf{grad}_{y^*} T_i^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)}) - k_a^*(\mathbf{grad}_{y^*} T_a^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)})) \cdot \mathbf{n_i} \tag{C3.20}$$

$$= L_{sg}^* \frac{D_v^*}{\rho_i^*}(\mathbf{grad}_{y^*} \rho_v^{*(1)} + \mathbf{grad}_{x^*} \rho_v^{*(0)}) \cdot \mathbf{n_i} \quad \text{on } \Gamma.$$

where the unknowns $T_i^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t)$ and $T_a^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t)$ are $\mathbf{y}^*$-periodic and the macroscopic gradient $\mathbf{grad}_{x^*} T^{*(0)}$ is given. Moreover we have the second-order problem for the equations (C3.3, C3.6) is written:

$$\mathrm{div}_{y^*}(D_v^*(\mathbf{grad}_{y^*} \rho_v^{*(1)} + \mathbf{grad}_{x^*} \rho_{vs}^{*(0)})) = 0 \quad \text{in } \Omega_a \tag{C3.21}$$

$$\rho_v^{*(1)} = \rho_{vs}^{*(1)} \quad \text{on } \Gamma. \tag{C3.22}$$

where the unknowns $\rho_v^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t)$ is $\mathbf{y}^*$-periodic. According to (A.40), this latter boundary condition can be also written

$$\rho_v^{*(1)} = \rho_{vs}^{*(1)} = \gamma^*(T^{*(0)}) T_a^{*(1)} \quad \text{on } \Gamma \tag{C3.23}$$

Moreover, we have

$$\mathbf{grad}_{x^*}\rho_{vs}^{*(0)} = \gamma^*(T^{*(0)})\mathbf{grad}_{x^*}T^{*(0)} \tag{C3.24}$$

thus equations (C3.21) and (C3.23) are written:

$$\mathrm{div}_{y^*}(D_v^*(\mathbf{grad}_{y^*}\rho_v^{*(1)} + \gamma^*(T^{*(0)})\mathbf{grad}_{x^*}T^{*(0)}) = 0 \quad \text{in } \Omega_a \tag{C3.25}$$

$$\rho_v^{*(1)} = \gamma^*(T^{*(0)})\,T_a^{*(1)} \quad \text{on } \Gamma \tag{C3.26}$$

As in the Cases C1 and C2, the solution of the above boundary value problems (C3.17-C3.20) and (C3.25-C3.26) appears as a linear function of the macroscopic gradient $\mathbf{grad}_{x^*}T^{*(0)}$, modulo an arbitrary function.

$$T_i^{*(1)}(\mathbf{x}^*,\mathbf{y}^*,t) = \mathbf{r}_i^*(\mathbf{y}^*)\cdot\mathbf{grad}_{x^*}T^{*(0)} + \widetilde{T}_i^{*(1)} \tag{C3.27}$$

$$T_a^{*(1)}(\mathbf{x}^*,\mathbf{y}^*,t) = \mathbf{r}_a^*(\mathbf{y}^*)\cdot\mathbf{grad}_{x^*}T^{*(0)} + \widetilde{T}_a^{*(1)} \tag{C3.28}$$

$$\rho_v^{*(1)}(\mathbf{x}^*,\mathbf{y}^*,t) = \gamma^*(T^{*(0)})\,(\mathbf{r}_a^*(\mathbf{y}^*)\cdot\mathbf{grad}_{x^*}T^{*(0)} + \widetilde{T}_a^{*(1)}) \quad \text{on } \Gamma \tag{C3.29}$$

where $\mathbf{r}_i^*(\mathbf{y}^*)$ and $\mathbf{r}_a^*(\mathbf{y}^*)$ are two periodic vectors which characterize the fluctuation of temperature in both phases at the pore scale. Introducing (C3.27) and (C3.28) in the set (C3.17-C3.20), these two vectors are solution of the following boundary value problem in a compact form:

$$\mathrm{div}_{y^*}(k_i^*(\mathbf{grad}_{y^*}\mathbf{r}_i^* + \mathbf{I})) = 0 \quad \text{in } \Omega_i \tag{C3.30}$$

$$\mathrm{div}_{y^*}(k_a^*(\mathbf{grad}_{y^*}\mathbf{r}_a^* + \mathbf{I})) = 0 \quad \text{in } \Omega_a \tag{C3.31}$$

$$\mathbf{r}_i^* = \mathbf{r}_a^* \quad \text{on } \Gamma \tag{C3.32}$$

$$(k_i^*(\mathbf{grad}_{y^*}\mathbf{r}_i^* + \mathbf{I}) - (k_a^* + L_{sg}^* D_v^* \frac{\gamma^*(T^{*(0)})}{\rho_i^*})(\mathbf{grad}_{y^*}\mathbf{r}_a^* + \mathbf{I}))\cdot\mathbf{n_i} = 0 \quad \text{on } \Gamma \tag{C3.33}$$

$$\frac{1}{|\Omega|}\int\limits_{\Omega}(\mathbf{r}_\mathrm{a}^* + \mathbf{r}_\mathrm{i}^*)\mathrm{d}\Omega = \mathbf{0} \tag{C3.34}$$

This latter equation is introduced to ensure the uniqueness of the solution. This latter boundary value problem is similar as the one of (A.20)-(A.24) where $k_a^*$ is now equal to $k_a^* + L_{sg}^* D_v^* \gamma^*(T^{*(0)})/\rho_i^*$. At the local scale, the thermal conductivity appears to be enhanced by the phase change.

#### S.3.3.2 Macroscopic description

Finally, the third order problem is given by Eq. (C3.1), (C3.2), (C3.4), and (C3.5) of order $\varepsilon^2$:

$$\rho_i^* C_i^* \frac{\partial T^{*(0)}}{\partial t^*} - \mathrm{div}_{y^*}(k_i^*(\mathbf{grad}_{y^*} T_i^{*(2)} + \mathbf{grad}_{x^*} T_i^{*(1)})) - \mathrm{div}_{x^*}(k_i^*(\mathbf{grad}_{y^*} T_i^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)})) = 0 \quad \text{in } \Omega_i \tag{C3.35}$$

$$\rho_a^* C_a^* \frac{\partial T^{*(0)}}{\partial t^*} - \mathrm{div}_{y^*}(k_a^*(\mathbf{grad}_{y^*} T_a^{*(2)} + \mathbf{grad}_{x^*} T_a^{*(1)})) - \mathrm{div}_{x^*}(k_a^*(\mathbf{grad}_{y^*} T_a^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)})) = 0 \quad \text{in } \Omega_a \tag{C3.36}$$

$$T_i^{*(2)} = T_a^{*(2)} \quad \text{on}\Gamma \tag{C3.37}$$

$$(k_i^*(\mathbf{grad}_{y^*} T_i^{*(2)} + \mathbf{grad}_{x^*} T_i^{*(1)}) - k_a^*(\mathbf{grad}_{y^*} T_a^{*(2)} + \mathbf{grad}_{x^*} T_a^{*(1)})) \cdot \mathbf{n_i} = \tag{C3.38}$$

$$= L_{sg}^* \frac{D_v^*}{\rho_i^*}(\mathbf{grad}_{y^*} \rho_v^{*(2)} + \mathbf{grad}_{x^*} \rho_v^{*(1)}) \cdot \mathbf{n_i} \quad \text{on } \Gamma.$$

where the unknowns $T_i^{*(2)}(\mathbf{x}^*, \mathbf{y}^*, t)$ and $T_a^{*(2)}(\mathbf{x}^*, \mathbf{y}^*, t)$ are $\mathbf{y}^*$-periodic. For the water vapor, the third order problem is given by Eq. (C3.3) and (C3.6) of order $\varepsilon^2$:

$$\frac{\partial \rho_{vs}^{*(0)}}{\partial t^*} - \mathrm{div}_{y^*}(D_v^*(\mathbf{grad}_{y^*} \rho_v^{*(2)} + \mathbf{grad}_{x^*} \rho_v^{*(1)})) - \mathrm{div}_{x^*}(D_v^*(\mathbf{grad}_{y^*} \rho_v^{*(1)} + \mathbf{grad}_{x^*} \rho_{vs}^{*(0)}(T^{*(0)}))) = 0 \quad \text{in } \Omega_a \tag{C3.39}$$

$$D_v^*(\mathbf{grad}_{y^*} \rho_v^{*(2)} + \mathbf{grad}_{x^*} \rho_v^{*(1)}) \cdot \mathbf{n_i} = \rho_i^* w_n^{*(4)} \quad \text{on } \Gamma \tag{C3.40}$$

Integrating (C3.35) over $\Omega_i$ and (C3.36) and (C3.39) over $\Omega_a$, and then using the divergence theorem, the periodicity condition, and the boundary conditions (B2.40) leads to the first order dimensionless description:

$$(\rho C)^{\mathrm{eff}*} \frac{\partial T^{*(0)}}{\partial t^*} - \mathrm{div}_{x^*}(\mathbf{k}^{\mathrm{td}*} \mathbf{grad}_{x^*} T^{*(0)}) = \int_{\Gamma} L_{sg}^* \frac{D_v^*}{\rho_i^*}(\mathbf{grad}_{y^*} \rho_v^{*(2)} + \mathbf{grad}_{x^*} \rho_v^{*(1)}) \cdot \mathbf{n_i} dS = -L_{sg}^* \dot{\phi}. \tag{C3.41}$$

where $(\rho C)^{\mathrm{eff}*}$ and $\mathbf{k}^{\mathrm{td}*}$ are the dimensionless effective thermal capacity and the apparent dimensionless thermal conductivity, respectively, defined as:

$$(\rho C)^{\mathrm{eff}*} = (1-\phi)\rho_i^* C_i^* + \phi \rho_a^* C_a^* \tag{C3.42}$$

$$\mathbf{k}^{\mathrm{td}*} = \frac{1}{|\Omega|}\left(\int_{\Omega_\mathrm{a}} k_a^*(\mathbf{grad}_{y^*} \mathbf{r}_a^*(\mathbf{y}^*) + \mathbf{I}) \mathrm{d}\Omega + \int_{\Omega_\mathrm{i}} k_i^*(\mathbf{grad}_{y^*} \mathbf{r}_i^*(\mathbf{y}^*) + \mathbf{I}) \mathrm{d}\Omega\right) \tag{C3.43}$$

where $\phi$ is the porosity. Integrating (C3.39) over $\Omega_a$, and then using the divergence theorem and the periodicity condition, leads to the first order dimensionless description:

$$\phi \frac{\partial \rho_{vs}^{*(0)}}{\partial t} - \mathrm{div}_{x^*}(\mathbf{D}^{\mathrm{td}*}\mathbf{grad}_{x^*}\rho_{vs}^{*(0)}(T^{*(0)})) = -\int\limits_{\Gamma} D_v^*(\mathbf{grad}_{y^*}\rho_v^{*(2)} + \mathbf{grad}_{x^*}\rho_v^{*(1)}) \cdot \mathbf{n_i} dS = \rho_i^* \dot{\phi}. \tag{C3.44}$$

where $\mathbf{D}^{\mathrm{td}*}$ is the apparent effective diffusion tensor defined as:

$$\mathbf{D}^{\mathrm{td}*} = \frac{1}{|\Omega|}\int\limits_{\Omega_a} D_v^*(\mathbf{grad}_{y^*}\ \mathbf{r}_a^*(\mathbf{y}^*) + \mathbf{I})\mathrm{d}\,\Omega \tag{C3.45}$$

## References

Auriault, J.-L., Boutin, C., and Geindreau., C.: Homogenization of coupled phenomena in heterogenous media, Wiley-ISTE, London, 2009.