Suplement of “Multiscale modeling of heat and mass transfer in dry snow: influence of the condensation coefficient and comparison with experiments”

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S.1 Case A

Taking into account of the order of magnitude of the dimensionless numbers, $[F^T_i] = \mathcal{O} \left([F^T_a]\right) = \mathcal{O} \left([F^o]\right) = \mathcal{O}(\varepsilon^2)$, $[K] = \mathcal{O}(1)$, $[H] = \mathcal{O}(\varepsilon^2)$, $[W_R] = \mathcal{O}(\varepsilon^2)$, the dimensionless microscopic description (13)-(18) becomes:

$$
\varepsilon^2 \rho^*_i C^*_i \frac{\partial T^*_i}{\partial t^*} - \text{div}^* \left(k^*_i \text{grad}^* T^*_i\right) = 0 \quad \text{in } \Omega_i \tag{A.1}
$$

$$
\varepsilon^2 \rho^*_a C^*_a \frac{\partial T^*_a}{\partial t^*} - \text{div}^* \left(k^*_a \text{grad}^* T^*_a\right) = 0 \quad \text{in } \Omega_a \tag{A.2}
$$

$$
\varepsilon^2 \frac{\partial \rho^*_v}{\partial t^*} - \text{div}^* \left(D^*_v \text{grad}^* \rho^*_v\right) = 0 \quad \text{in } \Omega_a \tag{A.3}
$$

$$
T^*_i = T^*_a \quad \text{on } \Gamma \tag{A.4}
$$

$$
k^*_i \text{grad}^* T^*_i \cdot \mathbf{n}_i - k^*_a \text{grad}^* T^*_a \cdot \mathbf{n}_i = \varepsilon^2 L_{sg}^* \mathbf{w}^* \cdot \mathbf{n}_i \quad \text{on } \Gamma \tag{A.5}
$$

$$
D^*_v \text{grad}^* \rho^*_v \cdot \mathbf{n}_i = \varepsilon^2 \rho^*_i \mathbf{w}^* \cdot \mathbf{n}_i \quad \text{on } \Gamma. \tag{A.6}
$$

This set of equations is completed by the Hertz-Knudsen equation (10) and the Clausius Clapeyron’s law (9) expressed in dimensionless form as:

$$
w^*_n = \mathbf{w}^* \cdot \mathbf{n}_i = \frac{\alpha^*_v}{\rho^*_i} w^*_n \left[\rho^*_v - \rho^*_v(T^*_a)\right] \quad \text{on } \Gamma \tag{A.7}
$$
\( \rho_{v^* a}(T^*_a) = \rho_{v^* a}^{\text{ref}*}(T^*_{\text{ref}*}) \exp \left[ \frac{L_{sg} m^*}{\rho^*_t k^*} \left( \frac{1}{T^*_{\text{ref}*}} - \frac{1}{T^*_a} \right) \right] \) \hspace{1cm} (A.8)

## S.1.1 Heat transfer

Introducing asymptotic expansions for \( T^*_{i} \) and \( T^*_{a} \) in the relations (A.1), (A.2), (A.4), (A.5) gives at the lowest order:

\[
\text{div}_y (k^*_i \text{grad}_y T^*_i(0)) = 0 \quad \text{in } \Omega_i \tag{A.9}
\]

\[
\text{div}_y (k^*_a \text{grad}_y T^*_a(0)) = 0 \quad \text{in } \Omega_a \tag{A.10}
\]

\[
T^*_{i(0)}(x^*,y^*,t) = T^*_{a(0)}(x^*,y^*,t) \quad \text{on } \Gamma \tag{A.11}
\]

\[
(k^*_i \text{grad}_y T^*_i(0) - k^*_a \text{grad}_y T^*_a(0)) \cdot n_i = 0 \quad \text{on } \Gamma \tag{A.12}
\]

where the unknowns \( T^*_i(x^*,y^*,t) \) and \( T^*_a(x^*,y^*,t) \) are \( y^* \)-periodic. It can be shown that the obvious solution of the above boundary value problem is given by:

\[
T^*_i(0) = T^*_a(0) = T^*(0)(x^*,t) \tag{A.13}
\]

At the first order, the temperature is independent of the microscopic dimensionless variable \( y^* \), i.e. we have only one temperature field. Taking into account of these results, equations (A.1), (A.2), (A.4), and (A.5) of order \( \varepsilon \) give the following second-order problem:

\[
\text{div}_y (k^*_i \text{grad}_y T^*_i(1) + \text{grad}_x T^*_i(0)) = 0 \quad \text{in } \Omega_i \tag{A.14}
\]

\[
\text{div}_y (k^*_a \text{grad}_y T^*_a(1) + \text{grad}_x T^*_a(0)) = 0 \quad \text{in } \Omega_a \tag{A.15}
\]

\[
T^*_i(1) = T^*_a(1) \quad \text{on } \Gamma \tag{A.16}
\]

\[
(k^*_i \text{grad}_y T^*_i(1) + \text{grad}_x T^*_i(0)) - k^*_a (\text{grad}_y T^*_a(1) + \text{grad}_x T^*_a(0)) \cdot n_i = 0 \quad \text{on } \Gamma \tag{A.17}
\]
where the unknowns $T^{*^{(1)}}_i(x^*, y^*, t)$ and $T^{*^{(1)}}_a(x^*, y^*, t)$ are $y^*$-periodic and the macroscopic gradient $\nabla_{x^*} T^{*^{(0)}}$ is given. The solution of the above boundary value problem appears as a linear function of the macroscopic gradient, modulo an arbitrary function $\tilde{T}^{*^{(1)}}(x^*, t)$:

$$T^{*^{(1)}}_i(x^*, y^*, t) = t^*_i(y^*) \cdot \nabla_{x^*} T^{*^{(0)}} + \tilde{T}^{*^{(1)}}_i \quad (A.18)$$

$$T^{*^{(1)}}_a(x^*, y^*, t) = t^*_a(y^*) \cdot \nabla_{x^*} T^{*^{(0)}} + \tilde{T}^{*^{(1)}}_a \quad (A.19)$$

where $t^*_i(y^*)$ and $t^*_a(y^*)$ are two periodic vectors which characterize the fluctuation of temperature in both phases at the pore scale. Introducing (A.18) and (A.19) in the set (A.14)-(A.17), these two vectors are solution of the following boundary value problem, expressed in a compact form as:

$$\text{div}_{y^*}(k^*_i(\nabla_{y^*} t^*_i + I)) = 0 \quad \text{in } \Omega_i \quad (A.20)$$

$$\text{div}_{y^*}(k^*_a(\nabla_{y^*} t^*_a + I)) = 0 \quad \text{in } \Omega_a \quad (A.21)$$

$$t^*_i = t^*_a \quad \text{on } \Gamma \quad (A.22)$$

$$k^*_i(\nabla_{y^*} t^*_i + I) - k^*_a(\nabla_{y^*} t^*_a + I) \cdot n_i = 0 \quad \text{on } \Gamma \quad (A.23)$$

$$\frac{1}{|\Omega|} \int_{\Omega} (t^*_a + t^*_i) d\Omega = 0 \quad (A.24)$$

This latter equation is introduced to ensure the uniqueness of the solution. Finally, the third order problem is given by the Eq. (A.1), (A.2), (A.4), and (A.5) of order ε^2:

$$\rho^*_i C^*_i \frac{\partial T^{*^{(0)}}}{\partial t^*} - \text{div}_{y^*}(k^*_i(\nabla_{y^*} T^{*^{(2)}}_i + \nabla_{x^*} T^{*^{(1)}}_i)) - \text{div}_{x^*}(k^*_i(\nabla_{y^*} T^{*^{(1)}}_i + \nabla_{x^*} T^{*^{(0)}})) = 0 \quad \text{in } \Omega_i \quad (A.25)$$

$$\rho^*_a C^*_a \frac{\partial T^{*^{(0)}}}{\partial t^*} - \text{div}_{y^*}(k^*_a(\nabla_{y^*} T^{*^{(2)}}_a + \nabla_{x^*} T^{*^{(1)}}_a)) - \text{div}_{x^*}(k^*_a(\nabla_{y^*} T^{*^{(1)}}_a + \nabla_{x^*} T^{*^{(0)}})) = 0 \quad \text{in } \Omega_a \quad (A.26)$$

$$T^{*^{(2)}}_i = T^{*^{(2)}}_a \quad \text{on } \Gamma \quad (A.27)$$
\( (k_i^* (\text{grad}_y T_i^*(2) + \text{grad}_x T_i^*(1)) - k_a^* (\text{grad}_y T_a^*(2) + \text{grad}_x T_a^*(1))) \cdot \mathbf{n}_1 = w^{(0)}_n \) on \( \Gamma \) \hspace{1cm} (A.28)

where the unknowns \( T_i^*(2) (x^*, y^*, t) \) and \( T_a^*(2) (x^*, y^*, t) \) are \( y^* \)-periodic and \( w^{(0)}_n \) is the normal interface velocity due to the sublimation-deposition process given, at the zero order, by the Hertz-Knudsen equation (A.7) and the Clausius Clapeyron’s law (A.8).

### S.1.2 Water vapor transfer

Introducing asymptotic expansions for \( \rho_v^* \) in the relations (A.3) and (A.6) gives at the lowest order:

\[ \text{div}_{y^*} (D_v^* \text{grad}_{y^*} \rho_v^{(0)}) = 0 \quad \text{in } \Omega_a \] \hspace{1cm} (A.29)

\[ D_v^* \text{grad}_{y^*} \rho_v^{(0)} \cdot \mathbf{n}_1 = 0 \quad \text{on } \Gamma. \] \hspace{1cm} (A.30)

where the unknown \( \rho_v^{(0)} (x^*, y^*, t) \) is \( y^* \)-periodic. It can be shown (Auriault et al., 2009) that the solution of the above boundary value problem is given by:

\[ \rho_v^{(0)} = \rho_v^{(0)} (x^*, t). \] \hspace{1cm} (A.31)

At the first order, the water vapor density is independent of the microscopic dimensionless variable \( y^* \). Taking into account of these results, the second-order problem is given by Eq. (A.3) and (A.6) of order \( \varepsilon \), which are:

\[ \text{div}_{y^*} (D_v^* (\text{grad}_{y^*} \rho_v^{(1)} + \text{grad}_{x^*} \rho_v^{(0)})) = 0 \quad \text{in } \Omega_a \] \hspace{1cm} (A.32)

\[ D_v^* (\text{grad}_{y^*} \rho_v^{(1)} + \text{grad}_{x^*} \rho_v^{(0)}) \cdot \mathbf{n}_1 = 0 \quad \text{on } \Gamma. \] \hspace{1cm} (A.33)

where the unknown \( \rho_v^{(1)} (x^*, y^*, t) \) is \( y^* \)-periodic and the macroscopic gradient \( \text{grad}_{x^*} \rho_v^{(0)} \) is given. The solution of the above boundary value problem appears as a linear function of the macroscopic gradient, modulo an arbitrary function \( \tilde{\rho}_v^{(1)} (x^*, t) \) (Auriault et al., 2009):

\[ \rho_v^{(1)} (x^*, y^*, t) = g_v^* (y^*) \cdot \text{grad}_{x^*} \rho_v^{(0)} + \tilde{\rho}_v^{(1)} (x^*, t) \] \hspace{1cm} (A.34)

where \( g_v^* (y^*) \) is a periodic vector which characterizes the fluctuation of water vapor density in the air phase at the pore scale.

Introducing (A.34) in the set (A.32)-(A.33), this vector is solution of the following boundary value problem, expressed in a compact form:

\[ \text{div}_{y^*} (D_v^* (\text{grad}_{y^*} g_v^* + \mathbf{I})) = 0 \quad \text{in } \Omega_a \] \hspace{1cm} (A.35)
\[ D^*_v (\text{grad}_y^* g_v^* + I) \cdot n_i = 0 \quad \text{on } \Gamma \quad (A.36) \]

\[ \frac{1}{|\Omega|} \int_{\Omega_a} g_v^* d\Omega = 0 \quad (A.37) \]

This latter equation is introduced to ensure the uniqueness of the solution. Finally, the third order problem is given by Eq. (A.3) and (A.6) of order \( \varepsilon^2 \):

\[ \frac{\partial \rho^{* (0)}}{\partial t^*} - \text{div}_y^* (D^*_v (\text{grad}_y^* \rho^{* (2)}_v + \text{grad}_x^* \rho^{* (1)}_v)) - \text{div}_x^* (D^*_v (\text{grad}_y^* \rho^{* (1)}_v + \text{grad}_x^* \rho^{* (0)}_v)) = 0 \quad \text{in } \Omega_a \quad (A.38) \]

where the unknown \( \rho^{* (2)}_v (x^*, y^*, t) \) is \( y^* \)-periodic and \( w^{* (0)}_n \) is the normal interface velocity due to the sublimation/deposition process given, at the zero order, by the Hertz-Knudsen equation (A.7) and the Clausius Clapeyron’s law (A.8). Taking into account the above results, we have:

\[ \rho^{* (0)}_{vs} (T^*_{a}) = \rho^{* (0)}_{vs} (T^{* \text{ref}}_{a}) \exp \left[ \frac{L_{sg}^* m^*}{\rho^*_i k^*} \left( \frac{1}{T^{* \text{ref}}} - \frac{1}{T^{* (0)}} \right) \right] \left( 1 + \varepsilon \frac{L_{sg}^* m^*}{\rho^*_i k^*} \frac{T_{a}^{* (1)}}{(T^{* (0)})^2} + \ldots \right) \quad (A.40) \]

This relation shows that the asymptotic development of the Clausius-Clapeyron’s law is written:

\[ \rho^{* (0)}_{vs} (T^*_{a}) = \rho^{* (0)}_{vs} (x^*, t) + \varepsilon \rho^{* (1)}_{vs} (x^*, y^*, t) + \ldots \quad (A.41) \]

where the first term \( \rho^{* (0)}_{vs} \), which depends on \( T^{* (0)} (x^*, t) \) only, is defined as:

\[ \rho^{* (0)}_{vs} (T^{* (0)}) = \rho^{* (0)}_{vs} (T^{* \text{ref}}_{a}) \exp \left[ \frac{L_{sg}^* m^*}{\rho^*_i k^*} \left( \frac{1}{T^{* \text{ref}}} - \frac{1}{T^{* (0)}} \right) \right] \quad (A.42) \]

The relation (A.42) shows that the normal velocity \( w^{* (0)}_n \) arising in the boundary condition (A.39) does not depend on \( y^* \).

From (A.7), \( w^{* (0)}_n \) is also written:

\[ w^{* (0)}_n = \frac{\alpha^*}{\rho^*_i} w^*_k \left[ \rho^{* (0)}_v - \rho^{* (0)}_{vs} (T^{* (0)}) \right] \quad (A.43) \]

### S.1.3 Macroscopic description

Integrating (A.25) over \( \Omega \), and (A.26) over \( \Omega_a \), and then using the divergence theorem, the periodicity condition, and the boundary conditions (A.28) leads to the first order dimensionless description:

\[ (\rho C)^{\text{eff}} \frac{\partial T^{* (0)}}{\partial t^*} - \text{div}_x^* (k^{\text{eff}} \text{grad}_x^* T^{* (0)}) = \text{SSA}_V L_{sg}^* w^{* (0)}_n \quad (A.44) \]
where \( \text{SSA}_V = |\Gamma| / |\Omega| \) is the specific surface area and where \((\rho C)^{\text{eff}*}\) and \(k^{\text{eff}*}\) are the dimensionless effective thermal capacity and the effective dimensionless thermal conductivity, respectively, defined as:

\[
(\rho C)^{\text{eff}*} = (1 - \phi)\rho_i^* C_i^* + \phi \rho_a^* C_a^*
\]

(A.45)

\[
k^{\text{eff}*} = \frac{1}{|\Omega|} \left( \int_{\Omega_i} k_i^*(\text{grad}_i \cdot t_i^*(y^*) + I) d\Omega + \int_{\Omega_a} k_a^*(\text{grad}_a \cdot t_a^*(y^*) + I) d\Omega \right)
\]

(A.46)

where \(\phi\) is the porosity. Consequently, integrating (A.38) over \(\Omega_a\), and then using the divergence theorem, the periodicity condition, and the boundary conditions (A.39) leads to the first order dimensionless description:

\[
\phi \frac{\partial \rho^{*(0)}}{\partial t} - \text{div}^* (D^{\text{eff}*} \text{grad}^* \rho^{*(0)}) = -\text{SSA}_V \rho_i^* w^{*(0)}
\]

(A.47)

where \(D^{\text{eff}*}\) is the dimensionless effective diffusion tensor defined as:

\[
D^{\text{eff}*} = \frac{1}{|\Omega|} \int_{\Omega_a} D_v^*(\text{grad} \cdot g_v^*(y^*) + I) d\Omega
\]

(A.48)

### S.2 Case B1 and B2

#### S.2.1 Case B1

Taking into account of the order of magnitude of the dimensionless numbers, \([F_T^i] = \mathcal{O}(\varepsilon^2)\), \([F_T^a] = \mathcal{O}(\varepsilon)\), \([F_T^g] = \mathcal{O}(1)\), \([\mathcal{H}] = \mathcal{O}(\varepsilon)\), \([\mathcal{W}_R] = \mathcal{O}(\varepsilon)\), the dimensionless microscopic description (13)-(18) becomes:

\[
\varepsilon^2 \rho_i^* C_i^* \frac{\partial T_i^*}{\partial t^*} - \text{div}^* (k_i^* \text{grad}^* T_i^*) = 0 \quad \text{in } \Omega_i
\]

(B1.1)

\[
\varepsilon^2 \rho_a^* C_a^* \frac{\partial T_a^*}{\partial t^*} - \text{div}^* (k_a^* \text{grad}^* T_a^*) = 0 \quad \text{in } \Omega_a
\]

(B1.2)

\[
\varepsilon^2 \frac{\partial \rho_v^*}{\partial t^*} - \text{div}^* (D_v^* \text{grad}^* \rho_v^*) = 0 \quad \text{in } \Omega_a
\]

(B1.3)

\[
T_i^* = T_a^* \quad \text{on } \Gamma
\]

(B1.4)

\[
k_i^* \text{grad}^* T_i^* \cdot n_i - k_a^* \text{grad}^* T_a^* \cdot n_i = \varepsilon L_s^g w^* \cdot n_i \quad \text{on } \Gamma
\]

(B1.5)
\[ D_v^* \text{grad}^* \rho_v^* \cdot n_i = \varepsilon \rho_v^* w^* \cdot n_i \quad \text{on } \Gamma. \] (B1.6)

This set of equations is completed by the Hertz-Knudsen equation (10) and the Clausius-Clapeyron’s law (9) expressed in dimensionless form as:

\[ w_n^* = w^* \cdot n_i = \frac{\alpha_i^*}{\rho_i^*} w_i^* [\rho_i^* - \rho_{vs}(T_i^*)] \quad \text{on } \Gamma \] (B1.7)

\[ \rho_{vs}(T_a^*) = \rho_{vs}^{ref*}(T_{ref}^*) \exp \left[ \frac{L_{sg}^* m_i^*}{\rho_i^* k_i^*} \left( \frac{1}{T_{ref}^*} - \frac{1}{T_a^*} \right) \right] \] (B1.8)

S.2.1.1 Heat transfer

Introducing asymptotic expansions for \( T_i^* \) and \( T_a^* \) in the relations (B1.1), (B1.2), (B1.4), and (B1.5) gives at the lowest order:

\[ \text{div}_y^* (k_i^* \text{grad}_y^* T_i^{* (0)}) = 0 \quad \text{in } \Omega_i \] (B1.9)

\[ \text{div}_y^* (k_a^* \text{grad}_y^* T_a^{* (0)}) = 0 \quad \text{in } \Omega_a \] (B1.10)

\[ T_i^{* (0)} = T_a^{* (0)} \quad \text{on } \Gamma \] (B1.11)

\[ (k_i^* \text{grad}_y^* T_i^{* (0)} - k_a^* \text{grad}_y^* T_a^{* (0)}) \cdot n_i = 0 \quad \text{on } \Gamma \] (B1.12)

where the unknowns \( T_i^{* (0)}(x^*, y^*, t) \) and \( T_a^{* (0)}(x^*, y^*, t) \) are \( y^* \)-periodic. It can be shown Auriault et al. (2009) that the obvious solution of the above boundary value problem is given by:

\[ T_i^{* (0)} = T_a^{* (0)} = T^{* (0)}(x^*, t). \] (B1.13)

At the first order, the temperature is independent of the microscopic dimensionless variable \( y^* \), i.e. we have only one temperature field. Taking into account of these results, Eq. (B1.1), (B1.2), (B1.4), and (B1.5) of order \( \varepsilon \) give the following second-order problem:

\[ \text{div}_y^* (k_i^* (\text{grad}_y^* T_i^{* (1)} + \text{grad}_x^* T^{* (0)})) = 0 \quad \text{in } \Omega_i \] (B1.14)

\[ \text{div}_y^* (k_a^* (\text{grad}_y^* T_a^{* (1)} + \text{grad}_x^* T^{* (0)})) = 0 \quad \text{in } \Omega_a \] (B1.15)
\( T_i^{\ast(1)} = T_a^{\ast(1)} \) on \( \Gamma \) \hspace{1cm} (B1.16)

\[
(k_i^* (\text{grad}_y, T_i^{\ast(1)} + \text{grad}_x, T^{\ast(0)}) - k_a^* (\text{grad}_y, T_a^{\ast(1)} + \text{grad}_x, T^{\ast(0)})) \cdot \mathbf{n}_i = L_{sg} w_n^{\ast(0)} \text{ on } \Gamma
\]

(B1.17)

where the unknowns \( T_i^{\ast(1)}(x^*,y^*,t) \) and \( T_a^{\ast(1)}(x^*,y^*,t) \) are \( y^* \)-periodic and the macroscopic gradient \( \text{grad}_x, T^{\ast(0)} \) is given.

Moreover, it can be shown that at the first order \( w_n^{\ast(0)} = 0 \) (see B1.37). As in the case A, the solution of the above boundary value problem appears as a linear function of the macroscopic gradient, modulo an arbitrary function \( \tilde{T}_i^{\ast(1)}(x^*,t) \) Auriault et al. (2009):

\[
T_i^{\ast(1)}(x^*,y^*,t) = t_i^* (y^*) \cdot \text{grad}_x, T^{\ast(0)} + \tilde{T}_i^{\ast(1)}
\]

(B1.18)

\[
T_a^{\ast(1)}(x^*,y^*,t) = t_a^* (y^*) \cdot \text{grad}_x, T^{\ast(0)} + \tilde{T}_a^{\ast(1)}
\]

(B1.19)

where \( t_i^* (y^*) \) and \( t_a^* (y^*) \) are two periodic vectors which characterize the fluctuation of temperature in both phases at the pore scale. Introducing (B1.18) and (B1.19) in the set (B1.14)-(B1.17), these two vectors are solution of the following boundary value problem in a compact form:

\[
\text{div}_y^* (k_i^* (\text{grad}_y, t_i^* + I)) = 0 \text{ in } \Omega_i
\]

(B1.20)

\[
\text{div}_y^* (k_a^* (\text{grad}_y, t_a^* + I)) = 0 \text{ in } \Omega_a
\]

(B1.21)

\[
t_i^* = t_a^* \text{ on } \Gamma
\]

(B1.22)

\[
\frac{1}{|\Omega|} \int_{\Omega} (t_a^* + t_i^*) d\Omega = 0
\]

(B1.24)

This latter equation is introduced to ensure the uniqueness of the solution. Finally, the third order problem is given by Eq. (B1.1), (B1.2), (B1.4), and (B1.5) of order \( \varepsilon^2 \):

\[
\rho_i^* C_i^* \frac{\partial T^{\ast(0)}}{\partial t^*} - \text{div}_y^* (k_i^* (\text{grad}_y, T_i^{\ast(2)} + \text{grad}_x, T_i^{\ast(1)})) - \text{div}_x^* (k_i^* (\text{grad}_y, T_i^{\ast(1)} + \text{grad}_x, T^{\ast(0)})) = 0 \text{ in } \Omega_i
\]

(B1.25)
\( \rho_a C_a \frac{\partial T^{* (0)}}{\partial t} - \text{div}_y (k_a (\text{grad}_y T^{* (2)} + \text{grad}_x T^{* (1)})) - \text{div}_x (k_a (\text{grad}_y T^{* (2)} + \text{grad}_x T^{* (1)})) = 0 \) in \( \Omega_a \) (B1.26)

\( T^{* (2)}_i = T^{* (2)}_a \) on \( \Gamma \) (B1.27)

\[
(k_i^* (\text{grad}_y T^{* (2)}_i + \text{grad}_x T^{* (1)}_i) - k_a^* (\text{grad}_y T^{* (2)} + \text{grad}_x T^{* (1)})) \cdot n_i = L^*_{sg} w^{* (1)}_n \text{ on } \Gamma
\]

where the unknowns \( T^{* (2)}_i (x^*, y^*, t) \) and \( T^{* (2)}_a (x^*, y^*, t) \) are \( y^* \)-periodic. Integrating (B1.25) over \( \Omega_i \) and (B1.26) over \( \Omega_a \), and then using the divergence theorem, the periodicity condition, and the boundary conditions (B1.28) leads to the first order dimensionless description:

\[
(\rho C)^{\text{eff}} \frac{\partial T^{* (0)}}{\partial t} - \text{div}_x (k^{\text{eff}} \text{grad}_x T^{* (0)}) = \int_{\Gamma} L^*_{sg} w^{* (1)}_n dS = -L^*_{sg} \dot{\phi}\]

where \( (\rho C)^{\text{eff}} \) and \( k^{\text{eff}} \) are the dimensionless effective thermal capacity and the effective dimensionless thermal conductivity respectively, defined, as in the Case A, by:

\[
(\rho C)^{\text{eff}} = (1 - \phi)\rho_i^* C_i^* + \phi \rho_a^* C_a^*
\]

\[
k^{\text{eff}} = \frac{1}{|\Omega|} \left( \int_{\Omega_a} k_a^* (\text{grad}_y t_a^* (y^*) + I) d\Omega + \int_{\Omega_i} k_i^* (\text{grad}_y t_i^* (y^*) + I) d\Omega \right)
\]

where \( \phi \) is the porosity.

**S.2.1.2 Water vapor transfer**

Introducing asymptotic expansions for \( \rho_v^* \) in the relations (B1.3) and (B1.6) gives at the lowest order:

\[
\text{div}_y (D_v^* \text{grad}_y \rho_v^{* (0)}) = 0 \text{ in } \Omega_a
\]

\[
D_v^* \text{grad}_y \rho_v^{* (0)} \cdot n_i = 0 \text{ on } \Gamma.
\]

where the unknown \( \rho_v^{* (0)} (x^*, y^*, t) \) is \( y^* \)-periodic. It can be shown(Auriault et al. (2009)) that the solution of the above boundary value problem is given by:

\[
\rho_v^{* (0)} = \rho_v^{* (0)} (x^*, t).
\]
At the first order, the water vapor density is independent of the microscopic dimensionless variable \( y^* \). Taking into account these results, the second-order problem is given by Eq. (B1.3) and (B1.6) of order \( \varepsilon \):

\[
\text{div}_{y^*}(D_v^*(\text{grad}_{y^*}\rho_v^{(1)} + \text{grad}_{x^*}\rho_v^{(0)})) = 0 \quad \text{in } \Omega_a \quad (B1.35)
\]

\[
D_v^*(\text{grad}_{y^*}\rho_v^{(1)} + \text{grad}_{x^*}\rho_v^{(0)}) \cdot \mathbf{n}_i = \alpha w_k \left[ \rho_v^{(0)} - \rho_{vs}^{(0)}(T^{(0)}) \right] \quad \text{on } \Gamma. \quad (B1.36)
\]

where the unknown \( \rho_v^{(1)}(x^*, y^*, t) \) is \( y^* \)-periodic. Consequently, integrating (B1.35) over \( \Omega_a \), and then using the divergence theorem, the periodicity condition, the boundary conditions (B1.36) and the result (B1.34) leads to the first order dimensionless description:

\[
\rho_v^{(0)} = \rho_{vs}^{(0)}(T^{(0)}) \quad (B1.37)
\]

Consequently, as in the Case A, the solution of the above boundary value problem (B1.35) - (B1.36) appears as a linear function of the macroscopic gradient \( \text{grad}_{x^*}\rho_{vs}^{(0)}(T^{(0)}) \) modulo an arbitrary function \( \tilde{\rho}_v^{(1)}(x^*, t) \):

\[
\rho_v^{(1)}(x^*, y^*, t) = g_v^*(y^*) \cdot \text{grad}_{x^*}\rho_{vs}^{(0)}(T^{(0)}) + \tilde{\rho}_v^{(1)}(x^*, t) \quad (B1.38)
\]

where \( g_v^*(y^*) \) is a periodic vector which characterizes the fluctuation of water vapor density in the air phase at the pore scale induced by the macroscopic gradient \( \text{grad}_{x^*}\rho_{vs}^{(0)}(T^{(0)}) \). Introducing (B1.38) in the set (B1.35)-(B1.36), this vector is solution of the following boundary value problem in a compact form:

\[
\text{div}_{y^*}(D_v^*(\text{grad}_{y^*}g_v^* + I)) = 0 \quad \text{in } \Omega_a \quad (B1.39)
\]

\[
D_v^*(\text{grad}_{y^*}g_v^* + I) \cdot \mathbf{n}_i = 0 \quad \text{on } \Gamma \quad (B1.40)
\]

\[
\frac{1}{|\Omega|} \int_{\Omega_a} g_v^* d\Omega = 0 \quad (B1.41)
\]

This latter equation is introduced to ensure the uniqueness of the solution. Finally, the third order problem is given by Eq. (B1.3) and (B1.6) of order \( \varepsilon^2 \):

\[
\frac{\partial \rho_{vs}^{(0)}}{\partial t^*} - \text{div}_{y^*}(D_v^*(\text{grad}_{y^*}\rho_v^{(2)} + \text{grad}_{x^*}\rho_v^{(1)})) - \text{div}_{x^*}(D_v^*(\text{grad}_{y^*}\rho_v^{(1)} + \text{grad}_{x^*}\rho_{vs}^{(0)}(T^{(0)}))) = 0 \quad \text{in } \Omega_i \quad (B1.42)
\]

\[
D_v^*(\text{grad}_{y^*}\rho_v^{(2)} + \text{grad}_{x^*}\rho_v^{(1)}) \cdot \mathbf{n}_i = \rho_i w_n^{(1)} \quad \text{on } \Gamma \quad (B1.43)
\]
\[\phi \frac{\partial \rho^{*\langle 0 \rangle}}{\partial t} - \text{div} \nu^{*} (D^{\text{eff}*} \nabla \rho^{*\langle 0 \rangle}) = \int_{\Gamma} \rho^{*\langle 1 \rangle} \cdot w^{*} \, dS = \rho^{*} \dot{\phi} \] (B1.44)

where \(D^{\text{eff}*}\) is the classical dimensionless effective diffusion tensor defined as (see Case A):

\[D^{\text{eff}*} = \frac{1}{|\Omega|} \int_{\Omega} D^{*} \nabla \nu^{*} \cdot \nabla \nu^{*} \, d\Omega\] (B1.45)

### S.2.2 Case B2

Taking into account of the order of magnitude of the dimensionless numbers, \([F_T] = \mathcal{O}(\left[F_T^a\right]) = \mathcal{O}(\left[F_T^h\right]) = \mathcal{O}(\varepsilon^2)\), \([K] = \mathcal{O}(1)\), \([H] = \mathcal{O}(1)\), \([W_R] = \mathcal{O}(1)\), the dimensionless microscopic description (13)-(18) becomes:

\[
\varepsilon^2 \rho^{*\langle i \rangle} \frac{\partial T^{*\langle i \rangle}}{\partial t} - \text{div}^* (k^{*\langle i \rangle} \nabla^* T^{*\langle i \rangle}) = 0 \quad \text{in} \ \Omega_i \tag{B2.1}
\]

\[
\varepsilon^2 \rho^{*\langle a \rangle} \frac{\partial T^{*\langle a \rangle}}{\partial t} - \text{div}^* (k^{*\langle a \rangle} \nabla^* T^{*\langle a \rangle}) = 0 \quad \text{in} \ \Omega_a \tag{B2.2}
\]

\[
\varepsilon^2 \frac{\partial \rho^{*\langle i \rangle}}{\partial t} - \text{div}^* (D^{\text{eff}*} \nabla \rho^{*\langle i \rangle}) = 0 \quad \text{in} \ \Omega_i \tag{B2.3}
\]

\[T^{*\langle i \rangle} = T^{*\langle a \rangle} \quad \text{on} \ \Gamma \tag{B2.4}
\]

\[k^{*\langle i \rangle} \nabla T^{*\langle i \rangle} \cdot \nu_i - k^{*\langle a \rangle} \nabla T^{*\langle a \rangle} \cdot \nu_i = L_{\text{sg}} \frac{D^{*}}{\rho^{*\langle i \rangle} \nu^{*\langle i \rangle}} \nabla \rho^{*\langle i \rangle} \cdot \nu_i \quad \text{on} \ \Gamma \tag{B2.5}
\]

\[D^{\text{eff}*} \nabla \rho^{*\langle i \rangle} \cdot \nu_i = \rho^{*\langle i \rangle} \nu^{*\langle i \rangle} \cdot \nu_i \quad \text{on} \ \Gamma. \tag{B2.6}
\]

This set of equations is completed by the Hertz-Knudsen equation (A.7) and the Clausius Clapeyron’s law (9) expressed in dimensionless form as:

\[w^{*\langle n \rangle} = \nu^{*\langle n \rangle} \cdot \nu_i = \frac{\alpha^{*}}{\rho^{*\langle i \rangle} \nu^{*\langle i \rangle}} L_{\text{sg}} m^{*} \left[\rho^{*\langle i \rangle} - \rho^{*\langle a \rangle} (T^{*\langle a \rangle})\right] \quad \text{on} \ \Gamma \tag{B2.7}
\]

\[\rho^{*\langle a \rangle} (T^{*\langle a \rangle}) = \rho^{*\text{ref}*\langle a \rangle} (T^{\text{ref}*}) \exp \left[\frac{L_{\text{sg}} m^{*}}{\rho^{*\langle i \rangle} \nu^{*\langle i \rangle} k^{*}} \left(\frac{1}{T^{\text{ref}*}} - \frac{1}{T^{*\langle a \rangle}}\right)\right] \tag{B2.8}
\]
S.2.2.1 Heat and water vapor transfer at the first order

Introducing asymptotic expansions for $T_i^*$ and $T_a^*$ in the relations (B2.1), (B2.2), (B2.4), and (B2.5) gives at the lowest order:

$$\text{div}_y(k_i^* \text{grad}_y T_i^{*(0)}) = 0 \quad \text{in } \Omega_i \quad (B2.9)$$

$$\text{div}_y(k_a^* \text{grad}_y T_a^{*(0)}) = 0 \quad \text{in } \Omega_a \quad (B2.10)$$

$$T_i^{*(0)} = T_a^{*(0)} \quad \text{on } \Gamma \quad (B2.11)$$

$$(k_i^* \text{grad}_y T_i^{*(0)} - k_a^* \text{grad}_y T_a^{*(0)}) \cdot n_i = \frac{L_{sg}}{\rho_i^*} D_v^* \text{grad}_y \rho_v^{*(0)} \cdot n_i \quad \text{on } \Gamma \quad (B2.12)$$

where the unknowns $T_i^{*(0)}(x^*, y^*, t)$ and $T_a^{*(0)}(x^*, y^*, t)$ are $y^*$-periodic. Introducing asymptotic expansions for $\rho_v^*$ in the relations (B2.3) and (B2.6) gives at the lowest order:

$$\text{div}_y(D_v^* \text{grad}_y \rho_v^{*(0)}) = 0 \quad \text{in } \Omega_a \quad (B2.13)$$

$$D_v^* \text{grad}_y \rho_v^{*(0)} \cdot n_i = \alpha^* \omega_k^* \left[ \rho_v^{*(0)} - \rho_{v_s}^{*(0)}(T^{*(0)}) \right] \quad \text{on } \Gamma. \quad (B2.14)$$

where the unknown $\rho_v^{*(0)}(x^*, y^*, t)$ is $y^*$-periodic. Consequently, integrating (B2.13) over $\Omega_a$, and then using the divergence theorem, the periodicity condition, the boundary conditions (B2.14) leads to:

$$\rho_v^{*(0)} = \rho_{v_s}^{*(0)} \quad \text{on } \Gamma. \quad (B2.15)$$

Taking into account this result, the solution of the above boundary value problem is given by:

$$T_i^{*(0)} = T_a^{*(0)} = T^{*(0)}(x^*, t). \quad (B2.16)$$

and

$$\rho_v^{*(0)} = \rho_v^{*(0)}(x^*, t) = \rho_{v_s}^{*(0)}(T^{*(0)}). \quad (B2.17)$$

At the first order, the temperature and the the water vapor density are independent of the microscopic dimensionless variable $y^*$, i.e. we have only one temperature field.
S.2.2.2 Heat and water vapor transfer at the second order

Taking into account of these results, Eq. (B2.1), (B2.2), (B2.4), and (B2.5) of order $\varepsilon$ give the following second-order problem:

$$\text{div}_y \left( k^*_i (\text{grad}_y T^*_i(1) + \text{grad}_x T^*_i(0)) \right) = 0 \quad \text{in } \Omega_i$$  \hspace{1cm} (B2.18)

$$\text{div}_y \left( k^*_a (\text{grad}_y T^*_a(1) + \text{grad}_x T^*_a(0)) \right) = 0 \quad \text{in } \Omega_a$$  \hspace{1cm} (B2.19)

$$T^*_i(1) = T^*_a(1) \quad \text{on } \Gamma$$  \hspace{1cm} (B2.20)

$$(k^*_i (\text{grad}_y T^*_i(1) + \text{grad}_x T^*_i(0)) - k^*_a (\text{grad}_y T^*_a(1) + \text{grad}_x T^*_a(0))) \cdot n_i = \frac{L^*}{\rho_i^*} D^*_v (\text{grad}_y \rho^*_v(1) + \text{grad}_x \rho^*_v(0)) \cdot n_i \quad \text{on } \Gamma \quad (B2.21)$$

where the unknowns $T^*_i(1)(x^*, y^*, t)$ and $T^*_a(1)(x^*, y^*, t)$ are $y^*$-periodic and the macroscopic gradient $\text{grad}_x T^*_i(0)$ is given.

The second-order problem for the water vapor is given by Eq. (B2.3) and (B2.6) of order $\varepsilon$:

$$\text{div}_y \left( D^*_v (\text{grad}_y \rho^*_v(1) + \text{grad}_x \rho^*_v(0)) \right) = 0 \quad \text{in } \Omega_a$$  \hspace{1cm} (B2.22)

$$D^*_v (\text{grad}_y \rho^*_v(1) + \text{grad}_x \rho^*_v(0)) \cdot n_i = \alpha^* w^*_h \left[ \rho^*_v(1) - \rho^*_v(0) \right] \quad \text{on } \Gamma \quad (B2.23)$$

where the unknowns $\rho^*_v(1)(x^*, y^*, t)$ is $y^*$-periodic. Consequently, integrating (B2.22) over $\Omega_a$, and then using the divergence theorem, the periodicity condition, the boundary conditions (B2.23) leads to:

$$\rho^*_v(1) = \rho^*_v(0) \quad \text{on } \Gamma.$$  \hspace{1cm} (B2.24)

This result also implies that

$$D^*_v (\text{grad}_y \rho^*_v(1) + \text{grad}_x \rho^*_v(0)) \cdot n_i = 0 \quad \text{on } \Gamma.$$  \hspace{1cm} (B2.25)

The above boundary value problem can not be solved by considering both boundary conditions (B2.24) and (B2.25) on the ice-air interface simultaneously. However, it seems reasonable to assume that in the range $\varepsilon^{1/2} \leq [W_R] \leq 1$ the Neumann boundary condition (B2.25) can be privileged (Case B2), whereas in the range $1 \leq [W_R] \leq \varepsilon^{-1/2}$ the Dirichlet boundary condition (B2.24) can be considered (Case C1). These two cases will give us the two extreme behaviours when $[W_R] = O(1)$.
By considering first the Neumann boundary condition (B2.25), the solution of the above boundary value problem for the temperature appears as a linear function of the macroscopic gradient, modulo an arbitrary function \( T^{\ast(1)}(x^\ast, t) \):

\[
T^{\ast(1)}_i(x^\ast, y^\ast, t) = t^*_i(y^\ast) \cdot \text{grad}_x T^{\ast(0)} + \tilde{T}^{\ast(1)}_i
\]  
(B2.26)

\[
T^{\ast(1)}_a(x^\ast, y^\ast, t) = t^*_a(y^\ast) \cdot \text{grad}_x T^{\ast(0)} + \tilde{T}^{\ast(1)}_a
\]  
(B2.27)

where \( t^*_i(y^\ast) \) and \( t^*_a(y^\ast) \) are two periodic vectors which characterize the fluctuation of temperature in both phases at the pore scale. Introducing (B2.26) and (B2.27) in the set (B2.18)-(B2.21), these two vectors are solution of the following boundary value problem in a compact form:

\[
\text{div}_y (k^\ast_i (\text{grad}_y t^*_i + I)) = 0 \quad \text{in } \Omega_i
\]  
(B2.28)

\[
\text{div}_y (k^\ast_a (\text{grad}_y t^*_a + I)) = 0 \quad \text{in } \Omega_a
\]  
(B2.29)

\[
t^*_i = t^*_a \quad \text{on } \Gamma
\]  
(B2.30)

\[
(k^\ast_i (\text{grad}_y t^*_i + I) - k^\ast_a (\text{grad}_y t^*_a + I)) \cdot n = 0 \quad \text{on } \Gamma
\]  
(B2.31)

\[
\frac{1}{|\Omega|} \int_\Omega (t^*_a + t^*_i) d\Omega = 0
\]  
(B2.32)

This latter equation is introduced to ensure the uniqueness of the solution. Similarly, solution of the boundary value problem (B2.22) and (B2.25) appears as a linear function of the macroscopic gradient \( \text{grad}_x \rho^{\ast(0)}_{vs} (T^{\ast(0)}) \) modulo an arbitrary function \( \tilde{\rho}^{\ast(1)}_v(x^\ast, t) \):

\[
\rho^{\ast(1)}_v(x^\ast, y^\ast, t) = g^*_v(y^\ast) \cdot \text{grad}_x \rho^{\ast(0)}_{vs} (T^{\ast(0)}) + \tilde{\rho}^{\ast(1)}_v(x^\ast, t)
\]  
(B2.33)

where \( g^*_v(y^\ast) \) is a periodic vector which characterizes the fluctuation of water vapor density in the air phase at the pore scale induced by the macroscopic gradient \( \text{grad}_x \rho^{\ast(0)}_{vs} (T^{\ast(0)}) \). Introducing (B2.33) in the set (B2.22)-(B2.25), this vector is solution of the following boundary value problem in a compact form:

\[
\text{div}_y (D^*_v (\text{grad}_y g^*_v + I)) = 0 \quad \text{in } \Omega_a
\]  
(B2.34)
\[ D^*(\text{grad}_y^* g^*_v + I) \cdot n_i = 0 \quad \text{on } \Gamma \] (B2.35)

\[ \frac{1}{|\Omega|} \int_{\Omega_i} g^*_v d\Omega = 0 \] (B2.36)

This latter equation is introduced to ensure the uniqueness of the solution.

### S.2.2.3 Macroscopic description

Finally, the third order problem for the heat transfer is given by Eq. (B2.1), (B2.2), (B2.4), and (B2.5) of order \( \epsilon^2 \):

\[ \rho_i^* C_i^* \frac{\partial T_i^{*(0)}}{\partial t^*} - \text{div}_y^* (k_i^* (\text{grad}_y^* T_i^{*(2)} + \text{grad}_x^* T_i^{*(1)})) - \text{div}_x^* (k_i^* (\text{grad}_y^* T_i^{*(1)} + \text{grad}_x^* T_i^{*(0)})) = 0 \quad \text{in } \Omega_i \] (B2.37)

\[ \rho_a^* C_a^* \frac{\partial T_a^{*(0)}}{\partial t^*} - \text{div}_y^* (k_a^* (\text{grad}_y^* T_a^{*(2)} + \text{grad}_x^* T_a^{*(1)})) - \text{div}_x^* (k_a^* (\text{grad}_y^* T_a^{*(1)} + \text{grad}_x^* T_a^{*(0)})) = 0 \quad \text{in } \Omega_a \] (B2.38)

\[ T_i^{*(2)} = T_a^{*(2)} \quad \text{on } \Gamma \] (B2.39)

\[ (k_i^* (\text{grad}_y^* T_i^{*(2)} + \text{grad}_x^* T_i^{*(1)}) - k_a^* (\text{grad}_y^* T_a^{*(2)} + \text{grad}_x^* T_a^{*(1)})) \cdot n_i = \frac{L^* s_g^*}{\rho_i^*} \int_{\text{grad}_v^* (\text{grad}_y^* \rho_i^{*(2)} + \text{grad}_x^* \rho_i^{*(1)})} \cdot n_i \quad \text{on } \Gamma \] (B2.40)

where the unknowns \( T_i^{*(2)}(x^*, y^*, t) \) and \( T_a^{*(2)}(x^*, y^*, t) \) are \( y^* \)-periodic. Integrating (B2.37) over \( \Omega_i \) and (B2.38) over \( \Omega_a \), and then using the divergence theorem, the periodicity condition, and the boundary conditions (B2.40) and the results leads to the first order dimensionless description:

\[ (\rho C)^{\text{eff}*} \frac{\partial T^{*(0)}}{\partial t^*} - \text{div}_x^* (k^{\text{eff}*} \text{grad}_x^* T^{*(0)}) = - \int_{\Gamma} L^* s_g^* w_n^{*(2)} dS = L^* \dot{\phi} \] (B2.41)

where \( (\rho C)^{\text{eff}*} \) and \( k^{\text{eff}*} \) are the dimensionless effective thermal capacity and the effective dimensionless thermal conductivity respectively, defined as:

\[ (\rho C)^{\text{eff}*} = (1 - \phi) \rho_i^* C_i^* + \phi \rho_a^* C_a^* \] (B2.42)

\[ k^{\text{eff}*} = \frac{1}{|\Omega|} \left( \int_{\Omega_i} k_i^* (\text{grad}_y^* t_i^* (y^*) + I) d\Omega + \int_{\Omega_a} k_a^* (\text{grad}_y^* t_a^* (y^*) + I) d\Omega \right) \] (B2.43)
where $\phi$ is the porosity.

Finally, the third order problem for the water vapor is given by Eq. (B2.3) and (B2.6) of order $\varepsilon^2$:

$$
\frac{\partial \rho^{(0)}_{i \ast} v_{i \ast}}{\partial t} - \text{div}_{y \ast} \left( D_y^{\ast} (\text{grad}_{y \ast} \rho^{(2)}_{i \ast} + \text{grad}_{x \ast} \rho^{(1)}_{i \ast}) \right) - \text{div}_{x \ast} \left( D_x^{\ast} (\text{grad}_{y \ast} \rho^{(1)}_{i \ast} + \text{grad}_{x \ast} \rho^{(0)}_{i \ast} (T^{(0)}_{i \ast})) \right) = 0 \quad \text{in } \Omega_i \quad (B2.44)
$$

$$
D_y^{\ast} (\text{grad}_{y \ast} \rho^{(2)}_{i \ast} + \text{grad}_{x \ast} \rho^{(1)}_{i \ast}) \cdot \mathbf{n} = \rho^*_{i \ast} w^* (2)_{i \ast} \quad \text{on } \Gamma \quad (B2.45)
$$

where the unknown $\rho^{(2)}_{i \ast} (x^*, y^*, t)$ is $y^*$-periodic and $w^* (2)_{i \ast}$ is the normal interface velocity due to the sublimation/deposition process at the first order. Consequently, integrating (B2.44) over $\Omega_i$, and then using the divergence theorem, the periodicity condition, and the boundary conditions (B2.45) leads to the first order dimensionless description:

$$
\phi \frac{\partial \rho^{(0)}_{i \ast} v_{i \ast}}{\partial t} - \text{div}_{x \ast} \left( \mathbf{D}_{\text{eff}}^{\ast} \text{grad}_{x \ast} \rho^{(0)}_{i \ast} (T^{(0)}_{i \ast}) \right) = \int_\Gamma \rho^*_{i \ast} w^* (2)_{i \ast} dS = \rho^*_{i \ast} \hat{\phi} \quad (B2.46)
$$

where $\mathbf{D}_{\text{eff}}^{\ast}$ is the classical dimensionless effective diffusion tensor defined as:

$$
\mathbf{D}_{\text{eff}}^{\ast} = \frac{1}{|\Omega|} \int_{\Omega_i} D_y^{\ast} (\text{grad}_{y \ast} \mathbf{g}_v(y^*) + \mathbf{I}) d \Omega \quad (B2.47)
$$

S.3 Cases C1, C2 et C3

S.3.1 Case C1

S.3.1.1 Heat and water vapor transfer at the second order

The Dirichlet boundary condition (B2.24) is now considered. According to (A.40), this boundary condition can be also written as:

$$
\rho^{(1)}_{i \ast} = \rho^{(1)}_{i \ast} v_{i \ast} = \gamma^* (T^{(0)}_{i \ast}) T^{(1)}_{a \ast} \quad \text{on } \Gamma \quad (C1.1)
$$

Moreover, we have

$$
\text{grad}_{x \ast} \rho^{(0)}_{i \ast} v_{i \ast} = \gamma^* (T^{(0)}_{i \ast}) \text{grad}_{x \ast} T^{(0)}_{i \ast} \quad (C1.2)
$$

thus Eq. (B2.22) and (B2.24) are written:

$$
\text{div}_{y \ast} \left( D_y^{\ast} (\text{grad}_{y \ast} \rho^{(1)}_{i \ast} + \gamma^* (T^{(0)}_{i \ast}) \text{grad}_{x \ast} T^{(0)}_{i \ast}) \right) = 0 \quad \text{in } \Omega_i \quad (C1.3)
$$

$$
\rho^{(1)}_{i \ast} = \gamma^* (T^{(0)}_{i \ast}) T^{(1)}_{a \ast} \quad \text{on } \Gamma \quad (C1.4)
$$
The solution of the boundary value problems (B2.18)-(B2.21) and (C1.3)-(C1.4) appears as a linear function of the macroscopic gradient $\nabla_x T^{*0}$, modulo an arbitrary function.

$$T^{*1}_i(x^*, y^*, t) = r^*_i(y^*) \cdot \nabla_x T^{*0} + \tilde{T}^{*1}_i$$  (C1.5)

$$T^{*1}_a(x^*, y^*, t) = r^*_a(y^*) \cdot \nabla_x T^{*0} + \tilde{T}^{*1}_a$$  (C1.6)

$$\rho_v^{*1}(x^*, y^*, t) = \gamma^*(T^{*0}) \cdot (r^*_a(y^*) \cdot \nabla_x T^{*0} + \tilde{T}^{*1}_a)$$  on $\Gamma$  (C1.7)

where $r^*_i(y^*)$ and $r^*_a(y^*)$ are two periodic vectors which characterize the fluctuation of temperature in both phases at the pore scale. Introducing (C1.5) and (C1.6) in the set (B2.18)-(B2.21) and (C1.3)-(C1.4), these two vectors are solution of the following boundary value problem in a compact form:

$$\text{div}_g^*(k^* (\nabla_g y^* r^*_i + I)) = 0 \quad \text{in } \Omega_i$$  (C1.8)

$$\text{div}_g^*((k^*_a + L_{sg} D_v^* \gamma^*(T^{*0}))/\rho^*_i)(\nabla_g y^* r^*_a + I) = 0 \quad \text{in } \Omega_a$$  (C1.9)

$$r^*_i = r^*_a \quad \text{on } \Gamma$$  (C1.10)

$$(k^*_i (\nabla_g y^* r^*_i + I) - (k^*_a + L_{sg} D_v^* \gamma^*(T^{*0}))/\rho^*_i)(\nabla_g y^* r^*_a + I) \cdot n = 0 \quad \text{on } \Gamma$$  (C1.11)

$$\frac{1}{|\Omega|} \int_{\Omega} (r^*_a + r^*_i) d\Omega = 0$$  (C1.12)

This latter equation is introduced to ensure the uniqueness of the solution. This latter boundary value problem is similar as the (A.20)-(A.24) where $k^*_a$ is now equal to $k^*_a + L_{sg} D_v^* \gamma^*(T^{*0})/\rho^*_i$. At the local scale, the thermal conductivity appears to be enhanced by the phase change.

### S3.1.2 Macroscopic description

Finally, the third order problem is given by Eq. (B2.1), (B2.2), (B2.4), and (B2.5) of order $\varepsilon^2$:

$$\rho^*_i C_i^* \frac{\partial T^{*0}}{\partial t} - \text{div}_y^* \left( k^*_i (\nabla_g y^* T^{*2}_i) + \nabla_x^* T^{*1}_i \right) - \text{div}_x^* \left( k^*_i (\nabla_g y^* T^{*1}_i + \nabla_x^* T^{*0}_i) \right) = 0 \quad \text{in } \Omega_i$$  (C1.13)
\[ \rho_a C_a \frac{\partial T_a^{*}}{\partial t} - \text{div}_y^*(k_a^*(\text{grad}_y^* T_a^{*2} + \text{grad}_x^* T_a^{*1})) - \text{div}_x^*(k_a^*(\text{grad}_y^* T_a^{*1} + \text{grad}_x^* T_a^{*0})) = 0 \quad \text{in} \ \Omega_a \quad (C1.14) \]

\[ T_i^{*2} = T_a^{*2} \quad \text{on} \Gamma \quad (C1.15) \]

\[ (k_a^*(\text{grad}_y^* T_i^{*2} + \text{grad}_x^* T_i^{*1}) - k_a^*(\text{grad}_y^* T_a^{*2} + \text{grad}_x^* T_a^{*1})) \cdot \mathbf{n}_i = \]

\[ = L_{sg}^* \frac{D_v^*}{\rho_t^*} (\text{grad}_y^* \rho_v^{*2} + \text{grad}_x^* \rho_v^{*1}) \cdot \mathbf{n}_i \quad \text{on} \Gamma. \quad (C1.16) \]

where the unknowns \( T_i^{*2} (x^*, y^*, t) \) and \( T_a^{*2} (x^*, y^*, t) \) are \( y^* \)-periodic. For the water vapor, the third order problem is given by Eq. (B2.3) and (B2.6) of order \( \varepsilon^2 \):

\[ \frac{\partial \rho_v^{*2}}{\partial t^*} - \text{div}_y^*(D_v^* (\text{grad}_y^* \rho_v^{*2} + \text{grad}_x^* \rho_v^{*1})) = \text{div}_x^*(D_v^* (\text{grad}_y^* \rho_v^{*1} + \text{grad}_x^* \rho_v^{*0}) (T^{*0})) = 0 \quad \text{in} \ \Omega_a \quad (C1.17) \]

\[ D_v^* (\text{grad}_y^* \rho_v^{*2} + \text{grad}_x^* \rho_v^{*1}) \cdot \mathbf{n}_i = \rho_i^* \rho_v^{*2} \quad \text{on} \Gamma \quad (C1.18) \]

Integrating (C1.13) over \( \Omega_i \) and (C1.14) over \( \Omega_a \), and then using the divergence theorem, the periodicity condition, and the boundary conditions (C1.16) leads to the first order dimensionless description:

\[ (\rho C)^{\text{eff}} \frac{\partial T^{*0}}{\partial t^*} - \text{div}_y^*(k^{\text{tds}} \text{grad}_x^* T^{*0}) = \int_{\Gamma} L_{sg}^* \frac{D_v^*}{\rho_t^*} (\text{grad}_y^* \rho_v^{*2} + \text{grad}_x^* \rho_v^{*1}) \cdot \mathbf{n}_i dS = -L_{sg}^* \dot{\phi}. \quad (C1.19) \]

where \( (\rho C)^{\text{eff}} \) and \( k^{\text{tds}} \) are the dimensionless effective thermal capacity and the apparent dimensionless thermal conductivity, respectively, defined as:

\[ (\rho C)^{\text{eff}} = (1 - \phi) \rho_t^* C_t^* + \phi \rho_a^* C_a^* \quad (C1.20) \]

\[ k^{\text{tds}} = \frac{1}{|\Omega|} \left( \int_{\Omega_a} k_a^*(\text{grad}_y^* r_i^*(y^*) + I) d\Omega + \int_{\Omega_i} k_i^*(\text{grad}_y^* r_i^*(y^*) + I) d\Omega \right) \quad (C1.21) \]

where \( \phi \) is the porosity. Integrating (C1.17) over \( \Omega_a \), and then using the divergence theorem and the periodicity condition, leads to the first order dimensionless description:

\[ \phi \frac{\partial \rho_v^{*2}}{\partial t} - \text{div}_x^*(D^{\text{tds}} \text{grad}_x^* \rho_v^{*0}) (T^{*0})) = - \int_{\Gamma} D_v^* (\text{grad}_y^* \rho_v^{*2} + \text{grad}_x^* \rho_v^{*1}) \cdot \mathbf{n}_i dS = \rho_i^* \dot{\phi}. \quad (C1.22) \]

where \( D^{\text{tds}} \) is the macroscopic effective diffusion tensor defined as:

\[ D^{\text{tds}} = \frac{1}{|\Omega|} \int_{\Omega_a} D_v^* (\text{grad}_y^* r_i^*(y^*) + I) d\Omega \quad (C1.23) \]
S.3.2 Case C2

Taking into account of the order of magnitude of the dimensionless numbers, \( [F^T_i] = \mathcal{O}( [F^T_a] ) = \mathcal{O}( [F^s] ) = \mathcal{O}( \varepsilon^2 ) \), \([K] = \mathcal{O}(1)\), \([W_R] = \mathcal{O}(\varepsilon^{-1})\), \([H] = \mathcal{O}(1)\), the dimensionless microscopic description (13)-(18) becomes:

\[
\varepsilon^2 \rho_i^* C_i^* \frac{\partial T_i^*}{\partial t^*} - \text{div}^* (k_i^* \text{grad}^* T_i^*) = 0 \quad \text{in } \Omega_i
\]  
(C2.1)

\[
\varepsilon^2 \rho_a^* C_a^* \frac{\partial T_a^*}{\partial t^*} - \text{div}^* (k_a^* \text{grad}^* T_a^*) = 0 \quad \text{in } \Omega_a
\]  
(C2.2)

\[
\varepsilon^2 \frac{\partial \rho_v^*}{\partial t^*} - \text{div}^* (D_v^* \text{grad}^* \rho_v^*) = 0 \quad \text{in } \Omega_a
\]  
(C2.3)

\[T_i^* = T_a^* \quad \text{on } \Gamma
\]  
(C2.4)

\[k_i^* \text{grad}^* T_i^* \cdot \mathbf{n}_i - k_a^* \text{grad}^* T_a^* \cdot \mathbf{n}_i = L_{sg}^* \frac{D_v^*}{\rho_i^*} \text{grad}^* \rho_v^* \cdot \mathbf{n}_i \quad \text{on } \Gamma
\]  
(C2.5)

\[D_v^* \text{grad}^* \rho_v^* \cdot \mathbf{n}_i = \varepsilon^{-1} \rho_i^* \mathbf{w}^* \cdot \mathbf{n}_i \quad \text{on } \Gamma.
\]  
(C2.6)

This set of equations is completed by the Hertz-Knudsen equation (A.7) and the Clausius Clapeyron’s law (9) expressed in dimensionless form as:

\[w_i^* = \mathbf{w}^* \cdot \mathbf{n}_i = \frac{\alpha_i^*}{\rho_i^*} w_k^* [\rho_v^* - \rho_v^*(T_a^*)] \quad \text{on } \Gamma
\]  
(C2.7)

\[\rho_v^*(T_a^*) = \rho_{v_s}^{\text{ref}}(T_a^{\text{ref}}) \exp \left[ \frac{L_{sg}^* m_v^*}{\rho_i^* k_s^*} \left( \frac{1}{T_a^{\text{ref}}} - \frac{1}{T_a^*} \right) \right]
\]  
(C2.8)

S.3.2.1 Heat transfer and water vapor transfer at the first and the second order

Introducing asymptotic expansions for \( T_i^* \) and \( T_a^* \) in the relations (C2.1), (C2.2), (C2.4), and (C2.5) give at the lowest order:

\[
\text{div}_y^* (k_i^* \text{grad}_y^* T_i^{*(0)}) = 0 \quad \text{in } \Omega_i
\]  
(C2.9)

\[
\text{div}_y^* (k_a^* \text{grad}_y^* T_a^{*(0)}) = 0 \quad \text{in } \Omega_a
\]  
(C2.10)
\[ T_i^{*0} = T_a^{*0} \quad \text{on } \Gamma \] (C2.11)

\[
\left( k_i^* \text{grad}_y^* T_i^{*0} - k_a^* \text{grad}_y^* T_a^{*0} \right) \cdot \mathbf{n}_i = L_{sg}^* \frac{D_v^*}{\rho_i^*} \text{grad}_y^* \rho_v^{*0} \quad \text{on } \Gamma
\] (C2.12)

where the unknowns \( T_i^{*0}(x^*, y^*, t) \) and \( T_a^{*0}(x^*, y^*, t) \) are \( y^* \)-periodic. Introducing asymptotic expansions for \( \rho_v^* \) in the relations (C2.3, C2.6) give at the lowest order:

\[ \text{div}_y^*(D_v^* \text{grad}_y^* \rho_v^{*0}) = 0 \quad \text{in } \Omega_a \] (C2.13)

\[ \rho_v^{*0} = \rho_{vs}^{*0}(T^{*0}) \quad \text{on } \Gamma. \] (C2.14)

where the unknown \( \rho_v^{*0}(x^*, y^*, t) \) is \( y^* \)-periodic. The solution of the above boundary value problems is given by:

\[ \rho_v^{*0} = \rho_{vs}^{*0}(x^*, t) = \rho_{vs}^{*0}(T^{*0}). \] (C2.15)

\[ T_i^{*0} = T_a^{*0} = T^{*0}(x^*, t). \] (C2.16)

At the first order, the temperature and the water vapor density are independent of the microscopic dimensionless variable \( y^* \). We have only one temperature field. Taking into account of these results, Eq. (C2.1), (C2.2), (C2.4), and (C2.5) of order \( \varepsilon \) give the following second-order problem:

\[ \text{div}_y^*(k_i^*(\text{grad}_y^* T_i^{*1} + \text{grad}_x^* T^{*0})) = 0 \quad \text{in } \Omega_i \] (C2.17)

\[ \text{div}_y^*(k_a^*(\text{grad}_y^* T_a^{*1} + \text{grad}_x^* T^{*0})) = 0 \quad \text{in } \Omega_a \] (C2.18)

\[ T_i^{*1} = T_a^{*1} \quad \text{on } \Gamma \] (C2.19)

\[
\left( k_i^*(\text{grad}_y^* T_i^{*1} + \text{grad}_x^* T^{*0}) - k_a^*(\text{grad}_y^* T_a^{*1} + \text{grad}_x^* T^{*0}) \right) \cdot \mathbf{n}_i = L_{sg}^* \frac{D_v^*}{\rho_i^*} (\text{grad}_y^* \rho_v^{*1} + \text{grad}_x^* \rho_v^{*0}) \cdot \mathbf{n}_i \quad \text{on } \Gamma.
\] (C2.20)
where the unknowns $T^*(1)(x^*, y^*, t)$ and $T^*_a(1)(x^*, y^*, t)$ are $y^*$-periodic and the macroscopic gradient $\text{grad}_{x^*} T^*(0)$ is given.

Moreover we have the second-order problem for Eq. (C2.3) and (C2.6) is written:

$$\text{div}_y (D^y_v (\text{grad}_y, \rho^*_v(1) + \text{grad}_{x^*} \rho^*_{vs}(0))) = 0 \quad \text{in } \Omega_a$$  \hspace{1cm} (C2.21)

$$\rho^*_v = \rho^*_{vs} \quad \text{on } \Gamma.$$  \hspace{1cm} (C2.22)

where the unknowns $\rho^*_v(1)(x^*, y^*, t)$ is $y^*$-periodic. According to (A.40), this latter boundary condition can be also written

$$\rho^*_v(1) = \gamma^* (T^*(0)) T^*_a(1) \quad \text{on } \Gamma$$  \hspace{1cm} (C2.23)

Moreover, we have

$$\text{grad}_{x^*} \rho^*_{vs}(0) = \gamma^* (T^*(0)) \text{grad}_{x^*} T^*(0)$$  \hspace{1cm} (C2.24)

thus Eq. (C2.21) and (C2.23) are written:

$$\text{div}_y (D^y_v (\text{grad}_y, \rho^*_v(1) + \gamma^* (T^*(0))\text{grad}_{x^*} T^*(0))) = 0 \quad \text{in } \Omega_a$$  \hspace{1cm} (C2.25)

$$\rho^*_v = \gamma^* (T^*(0)) T^*_a (\text{on } \Gamma)$$  \hspace{1cm} (C2.26)

As in the Case C1, the solution of the above boundary value problems (C2.17)-(C2.20) and (C2.25)-(C2.26) appears as a linear function of the macroscopic gradient $\text{grad}_{x^*} T^*(0)$, modulo an arbitrary function.

$$T^*(1)(x^*, y^*, t) = r^*_i(y^*) \cdot \text{grad}_{x^*} T^*(0) + \tilde{T}^*_i(1)$$  \hspace{1cm} (C2.27)

$$T^*_a(1)(x^*, y^*, t) = r^*_a(y^*) \cdot \text{grad}_{x^*} T^*(0) + \tilde{T}^*_a(1)$$  \hspace{1cm} (C2.28)

$$\rho^*_v(1)(x^*, y^*, t) = \gamma^* (T^*(0)) (r^*_a(y^*) \cdot \text{grad}_{x^*} T^*(0) + \tilde{T}^*_a(1)) \quad \text{on } \Gamma$$  \hspace{1cm} (C2.29)

where $r^*_i(y^*)$ and $r^*_a(y^*)$ are two periodic vectors which characterize the fluctuation of temperature in both phases at the pore scale. Introducing (C2.27) and (C2.28) in the set (C2.17)-(C2.20), these two vectors are solution of the following boundary value problem in a compact form:

$$\text{div}_y (k^*_i (\text{grad}_y, r^*_i + I)) = 0 \quad \text{in } \Omega_i$$  \hspace{1cm} (C2.30)
\[
\text{div}_y^*(k_a^*(\text{grad}_y^* r_a^* + \mathbf{I})) = 0 \quad \text{in } \Omega_a 
\] (C2.31)

\[
r_i^* = r_a^* \quad \text{on } \Gamma 
\] (C2.32)

\[
(k_i^* (\text{grad}_y^* r_i^* + \mathbf{I}) - (k_a^* + L_{sg} D_v^* \gamma^* (T_s(0))) (\text{grad}_y^* r_a^* + \mathbf{I})) \cdot \mathbf{n}_i = 0 \quad \text{on } \Gamma 
\] (C2.33)

\[
\frac{1}{|\Omega|} \int (r_a^* + r_i^*) d\Omega = 0 
\] (C2.34)

This latter equation is introduced to ensure the uniqueness of the solution. This latter boundary value problem is similar as the (A.20)-(A.24) where \(k_a^*\) is now equal to \(k_a^* + L_{sg} D_v^* \gamma^* (T_s(0))/\rho_i^*\). At the local scale, the thermal conductivity appears to be enhanced by the phase change.

520 S.3.2.2 Macroscopic description

Finally, the third order problem is given by the equations (C2.1, C2.2, C2.4, C2.5) of order \(\varepsilon^2\):

\[
\rho_i^* C_i^* \frac{\partial T_s(0)}{\partial t^*} - \text{div}_y^*(k_i^* (\text{grad}_y^* T_i(2) + \text{grad}_x^* T_i(1))) - \text{div}_x^*(k_i^* (\text{grad}_y^* T_i(2) + \text{grad}_x^* T_i(1))) = 0 \quad \text{in } \Omega_i 
\] (C2.35)

\[
\rho_a^* C_a^* \frac{\partial T_s(0)}{\partial t^*} - \text{div}_y^*(k_a^* (\text{grad}_y^* T_a(2) + \text{grad}_x^* T_a(1))) - \text{div}_x^*(k_a^* (\text{grad}_y^* T_a(2) + \text{grad}_x^* T_a(1))) = 0 \quad \text{in } \Omega_a 
\] (C2.36)

\[
T_i^{*}(2) = T_a^{*}(2) \quad \text{on } \Gamma 
\] (C2.37)

\[
(k_i^* (\text{grad}_y^* T_i^{*}(2) + \text{grad}_x^* T_i^{*}(1)) - k_a^* (\text{grad}_y^* T_a^{*}(2) + \text{grad}_x^* T_a^{*}(1)) \cdot \mathbf{n}_i = 
\] (C2.38)

\[
= L_{sg}^* D_v^* (\text{grad}_y^* \rho_v^{*}(2) \text{grad}_x^* \rho_v^{*}(1)) \cdot \mathbf{n}_i \quad \text{on } \Gamma.
\]

where the unknowns \(T_i^{*}(2)(x^*, y^*, t)\) and \(T_a^{*}(2)(x^*, y^*, t)\) are \(y^*\)-periodic. For the water vapor, the third order problem is given by the the equations (C2.3, C2.6) of order \(\varepsilon^2\):

\[
\frac{\partial \rho_v^{*(0)}}{\partial t^*} - \text{div}_y^* (D_v^*(\text{grad}_y^* \rho_v^{*(2)} \text{grad}_x^* \rho_v^{*(1)})) - \text{div}_x^* (D_v^*(\text{grad}_y^* \rho_v^{*(1)} \text{grad}_x^* \rho_v^{*(0)} (T_s(0)))) = 0 \quad \text{in } \Omega_a 
\] (C2.39)
 Integrating (C2.35) over \( \Omega_i \) and (C2.36) and (C2.39) over \( \Omega_a \), and then using the divergence theorem, the periodicity condition, and the boundary conditions (B2.40) leads to the first order dimensionless description:

\[
(\rho C)^{eff*} \frac{\partial T^{(0)}}{\partial t^*} - \text{div}_{x^*}(k^{td*} \text{grad}_{x^*} T^{(0)}) = \int_{\Gamma} L_{sg}^* \frac{D_v^*}{\rho_i^*} (\text{grad}_{y^*} \rho_v^{(2)} + \text{grad}_{x^*} \rho_v^{(1)}) \cdot n_i dS = -L_{sg}^* \dot{\phi}.
\]  

(C2.41)

where \((\rho C)^{eff*}\) and \(k^{td*}\) are the dimensionless effective thermal capacity and the apparent dimensionless conductivity respectively, defined as:

\[(\rho C)^{eff*} = (1 - \phi)\rho_i^* C_i^* + \phi \rho_a^* C_a^* \quad \text{(C2.42)}\]

\[k^{td*} = \frac{1}{|\Omega|} \left( \int_{\Omega_a} k_a^* (\text{grad}_{y^*} r_a^* (y^*) + I) d\Omega + \int_{\Omega_i} k_i^* (\text{grad}_{y^*} r_i^* (y^*) + I) d\Omega \right) \quad \text{(C2.43)}\]

where \(\phi\) is the porosity. Integrating (C2.39) over \( \Omega_a \), and then using the divergence theorem and the periodicity condition, leads to the first order dimensionless description:

\[\phi \frac{\partial \rho_v^{(0)}}{\partial t^*} - \text{div}_{x^*}(D^{td*} \text{grad}_{x^*} \rho_v^{(0)} (T^{(0)})) = -\int_{\Gamma} D_v^* (\text{grad}_{y^*} \rho_v^{(2)} + \text{grad}_{x^*} \rho_v^{(1)}) \cdot n_i dS = \rho_i^* \dot{\phi}. \quad \text{(C2.44)}\]

where \(D^{td*}\) is the apparent effective diffusion tensor defined as:

\[D^{td*} = \frac{1}{|\Omega|} \int_{\Omega_a} D_a^* (\text{grad}_{y^*} r_a^* (y^*) + I) d\Omega \quad \text{(C2.45)}\]

S.3.3 Case C3

Taking into account of the order of magnitude of the dimensionless numbers, \([F_i^T] = \mathcal{O}([F_a^T]) = \mathcal{O}([F_{sg}^\rho]) = \mathcal{O}(\varepsilon^2)\), \([K] = \mathcal{O}(1), [W_R] = \mathcal{O}(\varepsilon^{-1}), [H] = \mathcal{O}(1)\), the dimensionless microscopic description (13)-(18) becomes:

\[
\varepsilon^2 \rho_i^* C_i^* \frac{\partial T_i^*}{\partial t^*} - \text{div}^* (k_i^* \text{grad}^* T_i^*) = 0 \quad \text{in} \ \Omega_i \quad \text{(C3.1)}
\]

\[
\varepsilon^2 \rho_a^* C_a^* \frac{\partial T_a^*}{\partial t^*} - \text{div}^* (k_a^* \text{grad}^* T_a^*) = 0 \quad \text{in} \ \Omega_a \quad \text{(C3.2)}
\]

\[
\varepsilon^2 \frac{\partial \rho_v^*}{\partial t^*} - \text{div}^* (D_v^* \text{grad}^* \rho_v^*) = 0 \quad \text{in} \ \Omega_a \quad \text{(C3.3)}
\]
\[ T_i^* = T_a^* \quad \text{on} \, \Gamma \]  \hspace{1cm} (C3.4)

\[ k_i^* \text{grad}^* T_i^* \cdot n_i - k_a^* \text{grad}^* T_a^* \cdot n_i = L_{sg}^* \frac{D_v^*}{\rho_i^*} \text{grad}^* \rho_v^* \cdot n_i \quad \text{on} \, \Gamma \]  \hspace{1cm} (C3.5)

\[ D_v^* \text{grad}^* \rho_v^* \cdot n_i = \varepsilon^{-2} \rho_i^* \mathbf{w}^* \cdot n_i \quad \text{on} \, \Gamma . \]  \hspace{1cm} (C3.6)

This set of equations is completed by the Hertz-Knudsen equation (A.7) and the Clausius Clapeyron’s law (9) expressed in dimensionless form as:

\[ w_n^* = \mathbf{w}^* \cdot n_i = \frac{\alpha^*}{\rho_i^*} w_k^* [\rho_v^* - \rho_{vs}(T_a^*)] \quad \text{on} \, \Gamma \]  \hspace{1cm} (C3.7)

\[ \rho_{vs}(T_a^*) = \rho_{vs}^{ref*}(T^{ref*}) \exp \left[ \frac{L_{sg}^* m_i^*}{\rho_i^* h_i^*} \left( 1 - \frac{T^{ref*}}{T_a^*} \right) \right] \]  \hspace{1cm} (C3.8)

**S.3.3.1 Heat transfer and water vapor transfer at the first and second order**

Introducing asymptotic expansions for \( T_i^* \) and \( T_a^* \) in the relations (C3.1, C3.2, C3.4, C3.5) give at the lowest order:

\[ \text{div}_y^* (k_i^* \text{grad}_y^* T_i^{* (0)}) = 0 \quad \text{in} \, \Omega_i \]  \hspace{1cm} (C3.9)

\[ \text{div}_y^* (k_a^* \text{grad}_y^* T_a^{* (0)}) = 0 \quad \text{in} \, \Omega_a \]  \hspace{1cm} (C3.10)

\[ T_i^{* (0)} = T_a^{* (0)} \quad \text{on} \, \Gamma \]  \hspace{1cm} (C3.11)

\[ (k_i^* \text{grad}_y^* T_i^{* (0)} - k_a^* \text{grad}_y^* T_a^{* (0)}) \cdot n_i = L_{sg}^* \frac{D_v^*}{\rho_i^*} \text{grad}_y^* \rho_v^{* (0)} \quad \text{on} \, \Gamma \]  \hspace{1cm} (C3.12)

where the unknowns \( T_i^{* (0)}(x^*, y^*, t) \) and \( T_a^{* (0)}(x^*, y^*, t) \) are \( y^* \)-periodic. Introducing asymptotic expansions for \( \rho_v^* \) in the relations (C3.3, C3.6) give at the lowest order

\[ \text{div}_y^* (D_v^* \text{grad}_y^* \rho_v^{* (0)}) = 0 \quad \text{in} \, \Omega_a \]  \hspace{1cm} (C3.13)
\[ \rho_v^{*(0)} = \rho_{vs}^{*(0)}(T^{*(0)}) \quad \text{on } \Gamma. \]  \hspace{1cm} (C3.14)

where the unknowns \( \rho_v^{*(0)}(x^*, y^*, t) \) is \( y^* \)-periodic. The solution of the above boundary value problems is given by:

\[ \rho_v^{*(0)} = \rho_v^{*(0)}(x^*, t) = \rho_{vs}^{*(0)}(T^{*(0)}). \]  \hspace{1cm} (C3.15)

\[ T_i^{*(0)} = T_a^{*(0)} = T^{*(0)}(x^*, t). \]  \hspace{1cm} (C3.16)

At the first order, the temperature and the water vapor density are independent of the microscopic dimensionless variable \( y^* \). We have only one temperature field. Taking into account of these results, equations (C3.1, C3.2, C3.4, C3.5) of order \( \varepsilon \) give the following second-order problem:

\[ \text{div}_{y^*}(k_i^*(\text{grad}_{y^*} T_i^{*(1)} + \text{grad}_{x^*} T^{*(0)})) = 0 \quad \text{in } \Omega_i \]  \hspace{1cm} (C3.17)

\[ \text{div}_{y^*}(k_a^*(\text{grad}_{y^*} T_a^{*(1)} + \text{grad}_{x^*} T^{*(0)})) = 0 \quad \text{in } \Omega_a \]  \hspace{1cm} (C3.18)

\[ T_i^{*(1)} = T_a^{*(1)} \quad \text{on } \Gamma \]  \hspace{1cm} (C3.19)

\[ (k_i^*(\text{grad}_{y^*} T_i^{*(1)} + \text{grad}_{x^*} T^{*(0)}) - k_a^*(\text{grad}_{y^*} T_a^{*(1)} + \text{grad}_{x^*} T^{*(0)})) \cdot n_i = L^*_v \]  \hspace{1cm} (C3.20)

= \frac{D_v^*}{\rho_i^{*(1)}} \left( \text{grad}_{y^*} \rho_v^{*(1)} + \text{grad}_{x^*} \rho_{vs}^{*(0)} \right) \cdot n_i \quad \text{on } \Gamma.

where the unknowns \( T_i^{*(1)}(x^*, y^*, t) \) and \( T_a^{*(1)}(x^*, y^*, t) \) are \( y^* \)-periodic and the macroscopic gradient \( \text{grad}_{x^*} T^{*(0)} \) is given. Moreover we have the second-order problem for the equations (C3.3, C3.6) is written:

\[ \text{div}_{y^*}(D_v^*(\text{grad}_{y^*} \rho_v^{*(1)} + \text{grad}_{x^*} \rho_{vs}^{*(0)})) = 0 \quad \text{in } \Omega_a \]  \hspace{1cm} (C3.21)

\[ \rho_v^{*(1)} = \rho_{vs}^{*(1)} \quad \text{on } \Gamma. \]  \hspace{1cm} (C3.22)

where the unknowns \( \rho_v^{*(1)}(x^*, y^*, t) \) is \( y^* \)-periodic. According to (A.40), this latter boundary condition can be also written

\[ \rho_v^{*(1)} = \rho_{vs}^{*(1)} = \gamma^* (T^{*(0)}) T_a^{*(1)} \quad \text{on } \Gamma \]  \hspace{1cm} (C3.23)
Moreover, we have
\[ \nabla_x \rho_{\nu}^{(0)} = \gamma^* (T^*(0)) \nabla_x T^*(0) \] (C3.24)

thus equations (C3.21) and (C3.23) are written:
\[ \text{div}_y (D_v (\nabla_y \rho_{\nu}^{(1)} + \gamma^* (T^*(0)) \nabla_x T^*(0))) = 0 \quad \text{in } \Omega_a \] (C3.25)

\[ \rho_{\nu}^{(1)} = \gamma^* (T^*(0)) T_a^{(1)} \quad \text{on } \Gamma \] (C3.26)

As in the Cases C1 and C2, the solution of the above boundary value problems (C3.17-C3.20) and (C3.25-C3.26) appears as a linear function of the macroscopic gradient \( \nabla_x T^*(0) \), modulo an arbitrary function.

\[ T_i^{(1)}(x^*, y^*, t) = r_i^*(y^*) \cdot \nabla_x T^*(0) + \tilde{T}_i^{(1)} \] (C3.27)

\[ T_a^{(1)}(x^*, y^*, t) = r_a^*(y^*) \cdot \nabla_x T^*(0) + \tilde{T}_a^{(1)} \] (C3.28)

\[ \rho_{\nu}^{(1)}(x^*, y^*, t) = \gamma^* (T^*(0)) (r_a^*(y^*) \cdot \nabla_x T^*(0) + \tilde{T}_a^{(1)}) \quad \text{on } \Gamma \] (C3.29)

where \( r_i^*(y^*) \) and \( r_a^*(y^*) \) are two periodic vectors which characterize the fluctuation of temperature in both phases at the pore scale. Introducing (C3.27) and (C3.28) in the set (C3.17-C3.20), these two vectors are solution of the following boundary value problem in a compact form:

\[ \text{div}_y (k_i^* (\nabla_y r_i^* + I)) = 0 \quad \text{in } \Omega_i \] (C3.30)

\[ \text{div}_y (k_a^* (\nabla_y r_a^* + I)) = 0 \quad \text{in } \Omega_a \] (C3.31)

\[ r_i^* = r_a^* \quad \text{on } \Gamma \] (C3.32)

\[ (k_i^* (\nabla_y r_i^* + I) - (k_a^* + L_{sg} D_v \gamma^* (T^*(0)) \rho_i^*) (\nabla_y r_a^* + I)) \cdot n_i = 0 \quad \text{on } \Gamma \] (C3.33)

\[ \frac{1}{|\Omega|} \int_{\Omega} (r_a^* + r_i^*) d\Omega = 0 \] (C3.34)

This latter equation is introduced to ensure the uniqueness of the solution. This latter boundary value problem is similar as the one of (A.20)-(A.24) where \( k_a^* \) is now equal to \( k_a^* + L_{sg} D_v \gamma^* (T^*(0)) / \rho_i^* \). At the local scale, the thermal conductivity appears to be enhanced by the phase change.
Finally, the third order problem is given by Eq. (C3.1), (C3.2), (C3.4), and (C3.5) of order $\varepsilon^2$:

$$
\rho_i^s C_i^s \frac{\partial T_i^{s(0)}}{\partial t^s} - \text{div}_{\chi^s} \left( k_i^s (\text{grad}_{\chi^s} T_i^{s(2)} + \text{grad}_{x^s} T_i^{s(1)}) \right) - \text{div}_{x^s} \left( k_i^s (\text{grad}_{\chi^s} T_i^{s(1)} + \text{grad}_{x^s} T_i^{s(0)}) \right) = 0 \quad \text{in } \Omega_i \quad (C3.35)
$$

$$
\rho_a C_a^s \frac{\partial T_a^{s(0)}}{\partial t^s} - \text{div}_{\chi^s} \left( k_a^s (\text{grad}_{\chi^s} T_a^{s(2)} + \text{grad}_{x^s} T_a^{s(1)}) \right) - \text{div}_{x^s} \left( k_a^s (\text{grad}_{\chi^s} T_a^{s(1)} + \text{grad}_{x^s} T_a^{s(0)}) \right) = 0 \quad \text{in } \Omega_a \quad (C3.36)
$$

$$
T_i^{s(2)} = T_a^{s(2)} \quad \text{on } \Gamma \quad (C3.37)
$$

$$
(k_i^s (\text{grad}_{\chi^s} T_i^{s(2)} + \text{grad}_{x^s} T_i^{s(1)}) - k_a^s (\text{grad}_{\chi^s} T_a^{s(2)} + \text{grad}_{x^s} T_a^{s(1)}) \cdot n_i = L_a^s \frac{D_v^s}{\rho_i^s} \left( \text{grad}_{\chi^s} \rho_v^{s(2)} + \text{grad}_{x^s} \rho_v^{s(1)} \right) \cdot n_i \quad \text{on } \Gamma \quad (C3.38)
$$

where the unknowns $T_i^{s(2)}(x^s, y^s, t)$ and $T_a^{s(2)}(x^s, y^s, t)$ are $y^s$-periodic. For the water vapor, the third order problem is given by Eq. (C3.3) and (C3.6) of order $\varepsilon^2$:

$$
\frac{\partial \rho_v^{s(0)}}{\partial t^s} - \text{div}_{\chi^s} \left( D_v^s (\text{grad}_{\chi^s} \rho_v^{s(2)} + \text{grad}_{x^s} \rho_v^{s(1)}) \right) - \text{div}_{x^s} \left( D_v^s (\text{grad}_{\chi^s} \rho_v^{s(1)} + \text{grad}_{x^s} \rho_v^{s(0)} (T^{s(0)})) \right) = 0 \quad \text{in } \Omega_a \quad (C3.39)
$$

$$
D_v^s (\text{grad}_{\chi^s} \rho_v^{s(2)} + \text{grad}_{x^s} \rho_v^{s(1)}) \cdot n_i = \rho_i^s w^{s(4)} \quad \text{on } \Gamma \quad (C3.40)
$$

Integrating (C3.35) over $\Omega_i$ and (C3.36) and (C3.39) over $\Omega_a$, and then using the divergence theorem, the periodicity condition, and the boundary conditions (B2.40) leads to the first order dimensionless description:

$$
(\rho C)^{eff_s} \frac{\partial T^{s(0)}}{\partial t^s} - \text{div}_{x^s} (k^{tds} \text{grad}_{x^s} T^{s(0)}) = \int_{\Gamma} L_{sq}^s \frac{D_v^s}{\rho_i^s} \left( \text{grad}_{\chi^s} \rho_v^{s(2)} + \text{grad}_{x^s} \rho_v^{s(1)} \right) \cdot n_i dS = -L_{sq}^s \dot{\phi} \quad (C3.41)
$$

where $(\rho C)^{eff_s}$ and $k^{tds}$ are the dimensionless effective thermal capacity and the apparent dimensionless thermal conductivity, respectively, defined as:

$$(\rho C)^{eff_s} = (1 - \phi)\rho_i^s C_i^s + \phi \rho_a C_a^s \quad (C3.42)$$

$$
k^{tds} = \frac{1}{\Omega} \left( \int_{\Omega_a} k_a^s (\text{grad}_{\chi^s} r_a^s(y^s) + I)d\Omega + \int_{\Omega_i} k_i^s (\text{grad}_{\chi^s} r_i^s(y^s) + I)d\Omega \right) \quad (C3.43)
$$
where \( \phi \) is the porosity. Integrating (C3.39) over \( \Omega_a \), and then using the divergence theorem and the periodicity condition, leads to the first order dimensionless description:

\[
\phi \frac{\partial \rho^{* (0)}_v}{\partial t} - \text{div}_x \left( D^{td*} \text{grad}_x \rho^{* (0)}_v (T^{* (0)}) \right) = - \int_{\Gamma} D^*_v (\text{grad}_y \rho^{* (2)}_v + \text{grad}_x \rho^{* (1)}_v) \cdot \mathbf{n} dS = \rho^{* (0)}_v \phi.
\]  

(C3.44)

where \( D^{td*} \) is the apparent effective diffusion tensor defined as:

\[
D^{td*} = \frac{1}{|\Omega|} \int_{\Omega_a} D^*_v (\text{grad}_y \mathbf{r}^*_a (y^*) + \mathbf{I}) d\Omega
\]

(C3.45)
References