Review of
"Comment on: Macroscopic water vapor diffusion is not enhanced in snow" by Andrew C. Hansen

To simplify the reading, I will use Hansen's notations.

Two types of average

Assuming the existence of a Representative Elementary Volume $Y = Y_{ha} \cup Y_i$ where Y_{ha} and Y_i are respectively the volume occupied by the humid air and the ice phase, two types of average must be distinguished:

- the α-phase intrinsic average ($\alpha \in \{ha, i\}$)

$$\langle \bullet \rangle^\alpha = \frac{1}{|Y_\alpha|} \int_{Y_\alpha} \bullet \; \mathrm{d}V \tag{i}$$

- and the volume average

$$\langle \bullet \rangle = \frac{1}{|Y|} \int_Y \bullet \; \mathrm{d}V = \phi_{ha} \langle \bullet \rangle^{ha} + (1 - \phi_{ha}) \langle \bullet \rangle^i \tag{ii}$$

The paradox raised by Hansen comes from the difference between these two averages for the water vapor flow $\mathbf{J}_v$. As the water vapor flux is 0 in the ice phase

$$\langle \mathbf{J}_v \rangle = \phi_{ha} \langle \mathbf{J}_v \rangle^{ha} \tag{iii}$$

Homogenization in the general case

More than 30 years ago I worked on the heat and mass transfer in porous wet media [1, 8, 2]. If the solid surfaces of the porous material are wet, confounding the solid and liquid thermal conductivities, which are now the ice conductivity, the problem is exactly the same as the heat and mass transfer in snow. The case of wet porous media is easier to investigate experimentally measuring the macroscopic heat conductivity. More increasing the temperature leads to an increase of the contribution of the water vapor transport to the heat transfer helping to elucidate clearly the mechanisms.

We do both experiments [1] and the homogenization of the process using the volume averaging technique [8]. The main conclusions of this theoretical part are briefly recalled. The technical details are given in reference [2] unfortunately in french.

Two equations have to be solved at the snow macroscale corresponding to the water vapor mass balance and the heat transfer equation:

$$\phi_{ha} \frac{\partial \gamma_v}{\partial t} = \boldsymbol{\nabla} \cdot \left(\frac{\mathrm{d}\gamma_v}{\mathrm{d}\theta} \frac{\boldsymbol{D}}{1 - x_v} \cdot \boldsymbol{\nabla}\theta \right) - \dot{m} \tag{iva}$$

$$\langle \rho c \rangle \frac{\partial \theta}{\partial t} = \boldsymbol{\nabla} \cdot (\boldsymbol{k} \cdot \boldsymbol{\nabla}\theta) + u_{sg} \dot{m} \tag{ivb}$$

where

$\boldsymbol{k}$	is the Fourier thermal conductivity of the medium defined at the snow scale,
$\boldsymbol{\nabla}\theta$	$\equiv \langle\boldsymbol{\nabla}\theta\rangle$ is the volume average temperature gradient,
$\boldsymbol{D}$	is the macroscopic water vapor diffusion coefficient,
$\langle\rho c\rangle$	$= \phi_{ha}(\rho c)_{ha} + \phi_i(\rho c)_i$ is the volume heat capacity of the snow,
$\dot{m}$	is the mass of condensed water vapor into ice per unit volume of snow,
u_{sg}	the latent heat of sublimation.

In equation (iva) for the the vapor transport, note that the volume average of the water vapor flux is $\langle\mathbf{J}_v\rangle = -\dfrac{\mathrm{d}\gamma_v}{\mathrm{d}\theta}\dfrac{\boldsymbol{D}}{1-x_v}\cdot\boldsymbol{\nabla}\theta$, which defines the vapor diffusion coefficient (tensor) $\boldsymbol{D}$ at the macroscale for the snow. The factor $1-x_v$, where x_v is the vapor mole fraction in humid air in the denominator introduced by assuming stagnant air, can be discarded in the case of snow as the mole fraction x_v is small.

Combining the two preceding relations with the reasonable hypothesis that the transport and phase change terms are much larger than the temporal variation of the water vapor density leads to

$$\langle\rho c\rangle \frac{\partial\theta}{\partial t} = \boldsymbol{\nabla}\cdot(\boldsymbol{k}_{app}\cdot\boldsymbol{\nabla}\theta) \tag{v}$$

where the *apparent* thermal conductivity $\boldsymbol{k}_{app}$ is given by

$$\boldsymbol{k}_{app} = \boldsymbol{k} + u_{sg}\frac{\mathrm{d}\gamma_v}{\mathrm{d}\theta}\boldsymbol{D} = \boldsymbol{k} + k_{dif}\frac{\boldsymbol{D}}{\mathcal{D}_{v-a}} \tag{vi}$$

with $k_{dif} = u_{sg}\mathcal{D}_{v-a}\dfrac{\mathrm{d}\gamma_v}{\mathrm{d}\theta}$. The closure problem to be solved to determine the macroscopic coefficients is given for completeness in the Appendix. Starting from standard equations at the microscale the asymptotic homogenization process is a purely mathematical tool without any additional physical hypothesis. The homogenization process is similar to reference [3] except for the condition at the ice surface where thermodynamic equilibrium is imposed (air is saturated, which is called *fast kinetics hypothesis* in [5, 6]) rather than the impervious condition imposed by [3].

The normalized water vapor diffusion coefficient defined as $\mathbf{f} = \dfrac{\boldsymbol{D}}{\mathcal{D}_{v-a}}$ verifies the relation (see [2] and Appendix)

$$\phi_{ha}\langle\boldsymbol{\nabla}\theta\rangle^{ha} = \mathbf{f}\cdot\left(\phi_{ha}\langle\boldsymbol{\nabla}\theta\rangle^{ha} + \phi_i\langle\boldsymbol{\nabla}\theta\rangle^i\right) \tag{vii}$$

which proves, at least intuitively, that in the isotropic case f is less than 1 and that therefore the solid ice phase acts as a resistance to water vapor transport.

Historical note

The problem of simultaneous heat and mass transfer in wet porous media goes back to Krischer [7], who pointed out the important contribution of the evaporation-condensation process to the thermal conductivity noting that at temperature larger than 65 °C, $k_{dif} > k_{water}$. Therefore increasing water saturation, that is replacing k_{dif} by k_{water}, leads to a decrease of the apparent thermal conductivity. In the soil domain, the first contributors are Philip and de Vries [9] and de Vries [4]. Interestingly the unnecessary idea of an enhanced water vapor diffusion in porous media also exits in the soil domain: the fluid phase is supposed to transfer immediately the condensing water vapor due to a modification of the curvature of the meniscus (see figure 2 in [9]).

Layered microstructue

Application of homogenization to the particular case of a series medium (layered microstructure) has been considered in reference [2]:

$$k_{lm} = \frac{k_i \left(k_{ha} + (1 - \phi_{ha}) \, k_{dif}\right)}{(1 - \phi_{ha}) \left(k_{ha} + k_{dif}\right) + \phi_{ha} k_i} \tag{viiia}$$

$$f = \frac{D_{lm}}{\mathcal{D}_{v-a}} = \frac{\dfrac{\phi_{ha}}{k_{ha} + k_{dif}}}{\dfrac{1 - \phi_{ha}}{k_i} + \dfrac{\phi_{ha}}{k_{ha} + k_{dif}}} \leq 1 \tag{viiib}$$

$$k_{app} = k_{lm} + k_{dif} \frac{D_{lm}}{\mathcal{D}_{v-a}} = \frac{1}{\dfrac{1 - \phi_{ha}}{k_i} + \dfrac{\phi_{ha}}{k_{ha} + k_{dif}}} \tag{viiic}$$

The preceding result can be proved using simple arguments. First as the equivalent conductivity of the humid air phase is $k_{ha} + k_{dif}$, the apparent conductivity k_{app} of the layered medium is straightforwardly given by equation (viiic). Therefore the temperature gradient in the humid air is given by

$$(k_{ha} + k_{dif}) \left\langle \frac{\mathrm{d}\theta}{\mathrm{d}x} \right\rangle^{ha} = \frac{1}{\dfrac{1 - \phi_{ha}}{k_i} + \dfrac{\phi_{ha}}{k_{ha} + k_{dif}}} \left\langle \frac{\mathrm{d}\theta}{\mathrm{d}x} \right\rangle \tag{ix}$$

The water vapor flux in the humid air phase $\langle J_v \rangle^{ha}$ is given by

$$\langle J_v \rangle^{ha} = -\mathcal{D}_{v-a} \frac{\dfrac{1}{k_{ha} + k_{dif}}}{\dfrac{1 - \phi_{ha}}{k_i} + \dfrac{\phi_{ha}}{k_{ha} + k_{dif}}} \left\langle \frac{\mathrm{d}\theta}{\mathrm{d}x} \right\rangle \tag{x}$$

and the volume average water vapor flux is therefore given by

$$\langle J_v \rangle = \phi_{ha} \langle J_v \rangle^{ha} + (1 - \phi_{ha}) \langle J_v \rangle^{i} = \phi_{ha} \langle J_v \rangle^{ha} \tag{xi}$$

as the water vapor flux is 0 in the ice phase. Finally the normalized water vapor diffusion coefficient is given by

$$f = \frac{D_{lm}}{\mathcal{D}_{v-a}} = \frac{\dfrac{\phi_{ha}}{k_{ha} + k_{dif}}}{\dfrac{1 - \phi_{ha}}{k_i} + \dfrac{\phi_{ha}}{k_{ha} + k_{dif}}} \tag{xii}$$

The expression (viiia) for k_{lm} is simply deduced from the relation $k_{app} = k_{lm} + k_{dif} \dfrac{D_{lm}}{\mathcal{D}_{v-a}}$.

If the solid (ice) conductivity is much larger than the gas or the diffusive conductivity as

considered by Hansen, we have:

$$k_{lm} = \frac{k_{ha} + (1 - \phi_{ha})\, k_{dif}}{\phi_{ha}} \qquad \text{(xiiia)}$$

$$f = \frac{D_{lm}}{\mathcal{D}_{v-a}} = 1 \qquad \text{(xiiib)}$$

$$k_{app} = \frac{k_{ha} + k_{dif}}{\phi_{ha}} = \frac{k_{ha} + u_{sg}\mathcal{D}_{v-a}\dfrac{\mathrm{d}\gamma_v}{\mathrm{d}\theta}}{\phi_{ha}} \qquad \text{(xiiic)}$$

$$\text{(xiiid)}$$

These results are exactly those of [5, 6]. If in the decomposition of k_{app}, the heat conductivity only due to conduction k_0 is introduced

$$k_{app} = k_0 + f_{exp}\, k_{dif} \quad \text{with } k_0 = \frac{1}{\dfrac{\phi_{ha}}{k_{ha}} + \dfrac{(1 - \phi_{ha})}{k_i}} \qquad \text{(xiv)}$$

where the so-called experimental diffusion water diffusion factor f_{exp} (because easy to determine experimentally) is written as [1, 8]

$$f_{exp} = \frac{1}{\phi_{ha}\left(1 + \dfrac{\phi_i k_{ha}}{\phi_{ha} k_i}\right)\left(1 + \dfrac{\phi_{hi}(k_{ha} + k_{dif})}{\phi_{ha} k_i}\right)} \qquad \text{(xv)}$$

It is immediate to check that f_{exp} can exceed unity. In the limiting case of a very large value of k_i, i.e. when $(1 - \phi_{ha})/k_i \ll \phi_{ha}/(k_{ha} + k_{dif})$, the maximum value obtained is $f_{exp} \to 1/\phi_{ha}$.

These findings agree fully with Fourteau et al. [5, 6].

Comparison with Hansen's result

The table below summarizes Hansen's results and the result of the homogenization calculation.

	Hansen	Homogenization
k_{lm}	$\dfrac{k_{ha}}{\phi_{ha}}$	$\dfrac{k_{ha} + (1 - \phi_{ha})\mathcal{D}_{v-a}u_{sg}\dfrac{\mathrm{d}\gamma_v}{\mathrm{d}\theta}}{\phi_{ha}}$
D_{lm}	$\dfrac{\mathcal{D}_{v-a}}{\phi_{ha}}$	$\mathcal{D}_{v-a}$
$k_{app} = k_{lm} + D_{lm}u_{sg}\dfrac{\mathrm{d}\gamma_v}{\mathrm{d}\theta}$	$\dfrac{k_{ha} + \mathcal{D}_{v-a}u_{sg}\dfrac{\mathrm{d}\gamma_v}{\mathrm{d}\theta}}{\phi_{ha}}$	idem

If the definition (11) proposed by Hansen of the energy flux is correct, the splitting between the conductive flux (12) and the mass flux (13) does not give the rigorous result furnished by the homogenization procedure.

Definition of the macroscopic diffusion coefficient or the thermal conductivity

It should be noted that the definition of the macroscopic diffusion coefficient or the thermal conductivity has nothing to do with the modification of the snow structure. The definition of these coefficients implies that, when subjected to a concentration or a temperature gradient, the medium reacts *instantaneously* by a mass or heat flux. Since the ice front velocity is much lower than the vapor velocity, the medium can be assumed motionless. The temporal rearranging of the snow is another problem.

Other comments

• Line 299-303
The diffusion coefficient (16) obtained also by Fourteau et al. [5] is defined to represent the volume average water vapor flux $\langle \mathbf{J}_v \rangle$, which is needed in the macroscopic equations (iv). The flux leaving the upper boundary condition is a microscopic flux in the humid air phase, which in the particular case of the layered medium is equal to the humid air phase average: $\langle \mathbf{J}_v \rangle^{ha} = \langle \mathbf{J}_v \rangle / \phi_{ha}$.

 • Line 305-309
The Fourier thermal conductivity k_{lm} defined by (xiiia) obviously contains a *diffusive part*. If not, using $k_{app} = k_0 + f_{exp}\lambda_{dif}$ can lead to f_{exp} values larger than 1, which is not expected.
 • §3.4
At the homogenized macroscale, it is no more possible to distinguish separately the contribution of the two phases. This is evident when thinking about more complex geometries than the layered microstructure.

Conclusion

I completely agree with the calculations and arguments of Fourteau et al. [5, 6]. Hansen's calculations in the simple case of a multilayer structure are quite correct considering the internal modification of the snow structure. But his definition of the macroscopic coefficients (the splitting between the conductive and the diffusive parts) does not coincide with the precise rules given by the scaling process.

References

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Appendix: closure problem and definition of the macroscopic coefficients

Assuming the existence of a Representative Elementary Volume, the different quantities can be computed using a so-called closure problem on a spatially periodic unit cell $Y = Y_{ha} \cup Y_i$ for the function $\boldsymbol{\chi}_\alpha$ ($\alpha \in \{ha, i\}$) with periodic boundary conditions at the external cell frontiers Y:

$$Y_{ha} \quad : \quad \nabla^2 \boldsymbol{\chi}_{ha} = 0 \qquad \text{(xvia)}$$

$$Y_i \quad : \quad \nabla^2 \boldsymbol{\chi}_i = 0 \qquad \text{(xvib)}$$

$$\partial Y_{gs} \quad : \quad \boldsymbol{\chi}_{ha} = \boldsymbol{\chi}_i \qquad \text{(xvic)}$$

$$-\mathbf{n} \cdot \left[(k_{ha} + k_{dif}) \left(\boldsymbol{I} + \boldsymbol{\nabla} \boldsymbol{\chi}_{ha} \right) \right] = -\mathbf{n} \cdot k_i \left(\boldsymbol{I} + \boldsymbol{\nabla} \boldsymbol{\chi}_i \right) \qquad \text{(xvid)}$$

Therefore the normalized water vapor diffusion coefficient is given by:

$$\mathbf{f} = \frac{\boldsymbol{D}}{\mathcal{D}_{v-a}} \quad = \quad \phi_{ha} \left(\boldsymbol{I} + \langle \boldsymbol{\nabla} \boldsymbol{\chi}_{ha} \rangle^{ha} \right) \qquad \text{(xvii)}$$

where $\boldsymbol{I}$ is the identity tensor. The Fourier thermal conductivity is given by

$$\boldsymbol{k} \quad = \quad (1 - \phi_{ha}) \, k_i \left(\boldsymbol{I} + \langle \boldsymbol{\nabla} \boldsymbol{\chi}_i \rangle^i \right) + \phi_{ha} k_{ha} \left(\boldsymbol{I} + \langle \boldsymbol{\nabla} \boldsymbol{\chi}_{ha} \rangle^{ha} \right) \qquad \text{(xviii)}$$

and the apparent thermal conductivity is given by:

$$\boldsymbol{k}_{app} \quad = \quad \boldsymbol{k} + u_{sg} \frac{\mathrm{d}\gamma_v}{\mathrm{d}\theta} \mathcal{D}_{v-a} \phi_{ha} \left(\boldsymbol{I} + \langle \boldsymbol{\nabla} \boldsymbol{\chi}_{ha} \rangle^{ha} \right) \qquad \text{(xix)}$$

It is immediate to verify that the apparent thermal conductivity of the medium $\boldsymbol{k}_{app}$ is that of a two-phase medium comprising a solid phase of conductivity k_i and a gas phase of conductivity $k_{ha} + k_{dif}$.

Finally note that [2]

$$\langle \boldsymbol{\nabla} \theta_{ha} \rangle^{ha} = \left(\boldsymbol{I} + \langle \boldsymbol{\nabla} \boldsymbol{\chi}_{ha} \rangle^{ha} \right) \cdot \boldsymbol{\nabla} \theta \qquad \text{(xx)}$$