Referee's comments are in blue, our reply in black, quotes in the revised manuscript in purple.

This paper describes a method that can be used to evaluate observations of geothermal heat flux, and its application to a large region of Antarctica: the Lambert glacier and its drainage basin. A sophisticated model is used to estimate warm bedded regions given observations of ice geometry and surface velocity and several estimates of the geothermal heat flux field (GHF). The resulting warm bedded regions vary considerably depending on the choice of GHF, allowing the paper to rank them by comparing those regions to known locations of sub-glacial lakes. The Lambert glacier is representative of the majority of Antarctica since it is large and includes cold based ice, warm based ice that sees slow sliding, and warm based ice that sees fast sliding. That suggest that the method could be applied more widely, so both the method and its results should be of interest. The paper is generally well written and clear.

Thanks for your encouraging comments.

General Comments

The ice flow model (a Stokes flow model) is a complex one. It is certainly a better choice given unlimited resources than any of its common approximations (SIA, SSA, HOM…) and looks to have been applied correctly, but why is it necessary in this case? In which parts of the domain? It seems that in some parts you only use the direction from Elmer/Ice: how much does that differ from the direction of the surface gradient? The discussion says that this work 'builds on the earlier inversion method employed by Wolovick et al'. (which is SIA based) but how important is that extra effort?

Reply: Unlike Wolovick et al. (2021), we can not use SIA because we have a fast-flowing glacier and a floating shelf in our domain, and SIA does not represent those regimes well. Conversely, we cannot use SSA because we would also like to have an accurate solution in slow-flowing interior areas that move by internal deformation, and SSA does not represent internal deformation. We could have used a Higher Order model instead of full stokes, however, Elmer/Ice does not have an option for HOM. Therefore, our only option was full stokes.

Our description is incomplete. The surface velocity actually has 3 sources: the direction of surface gradient, Elmer/Ice modelled velocity and observations. The observations are used where flow is fast, Elmer/Ice modelled velocity is used where flow is slow, and the surface gradient is only used near the margins of the domain where the Elmer/Ice modelled velocity is not reliable. We add these descriptions in the revision.

We compared the direction from surface gradient, Elmer/Ice modelled velocity, and the observed velocity direction, see the figure below. As shown, there is large difference between modelled and observed velocity in the slow flow region. The Elmer/Ice model gives a better velocity field, and it is important because we need to use the modelled basal velocity and basal shear force to calculate the basal friction heat and basal melting rate. We add this plot in section 3.3 of the revision.

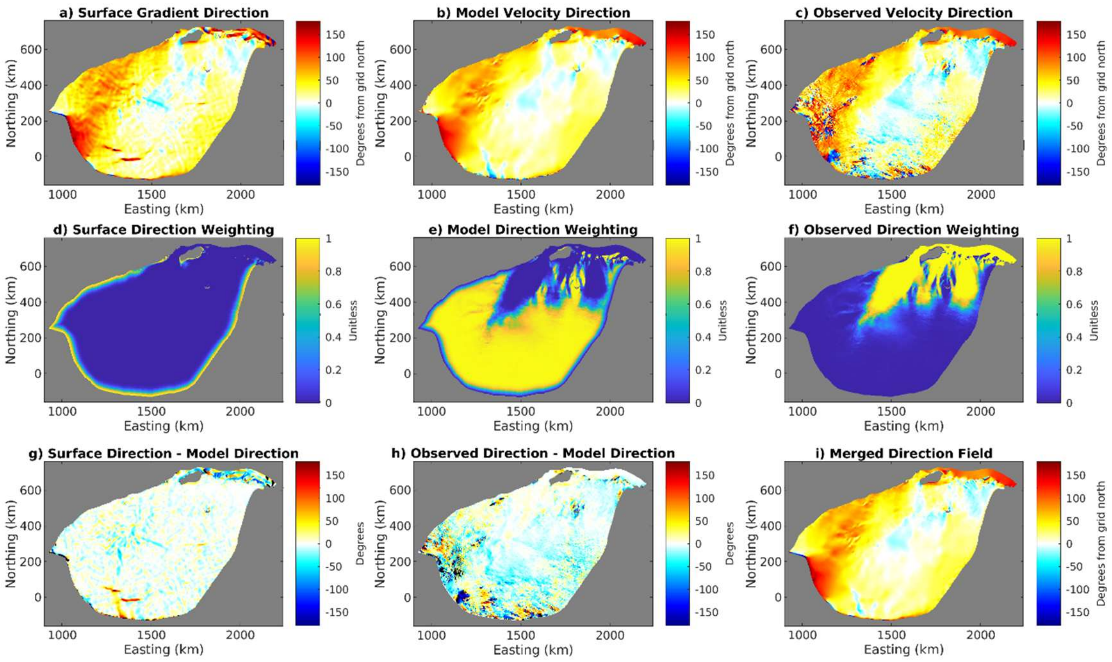


Figure. Velocity direction fields, in degrees clockwise from grid north. The first row shows the direction from surface gradient, Elmer/Ice modelled velocity, and the observed velocity direction. The middle row shows the 3 weighting fields (the sum of these weights is 1). The bottom row shows the difference between the direction of surface gradient and Elmer/Ice modelled velocity (plot g), the difference between the observed velocity direction and Elmer/Ice modelled velocity (plot h), and the merged velocity field used in the forward model (plot i).

Equation 4 appears to be a ‘cold ice’ model, i.e one that assumes T < Tm. The manuscript should justify this choice, with reference to polythermal models e.g Aschwanden 2013.

Reply: Yes, the modelled ice temperature is subject to the condition $T \le T_m$. Another referee said “The section of 3.1.1, 3.1.2, 3.1.3 is nearly same with Wolovick et al. (2021). The authors could just cite this paper rather than copy all these sections. Just make it clear about the different setup you used from Wolovick et al. (2021).” Therefore, we removed the descriptive text on the same setup as used from Wolovick et al. (2021) in the revision, including this Equation 4.

Use consistent notation for vectors etc throughout.

Reply: Done.

Specific Comments

L17. Are abbreviations (GHF) permitted in the abstract?

Reply: We think so but we are not sure. We can change if the editor says no.

L38. "Suggesting.."? How?

Reply: Rewritten as: However, there is also evidence of extensive subglacial rifts and lakes (Fretwell et al., 2013; Jamieson et al., 2016; Cui et al., 2020a). Jamieson et al. (2016) report a large subglacial drainage network in Princess Elizabeth Land (PEL), which would transport water from central PEL toward the Lambert-Amery region. The complexity of subglacial environment may influence the stability and basal mass balance of this area.

L41 ice penetrating radar *data*

Reply：Done.

L50, infers -> implies?

Reply: Done.

L67. comments on melt-water routing seem out of place in this paragraph

Reply: We move this sentence to an earlier location in this paragraph. Then it is "Ice at the melting point can lead to water, flowing along hydraulic gradients, and accumulating in local depressions (Fricker et al., 2016). The meltwater lubricates the ice/bed interface or saturates any sediment till layer and facilitates higher ice velocities via basal sliding."

L73. 'Ice sheet models are useful tools' is a matter of opinion, and not connected to the rest of the paragraph.

Reply: We change it to "Ice sheet models can be used to simulate the dynamics and thermodynamics of the ice sheet", and move it to the beginning of the next paragraph.

L97 "Hence, we make inferences" –> We state / We determine?

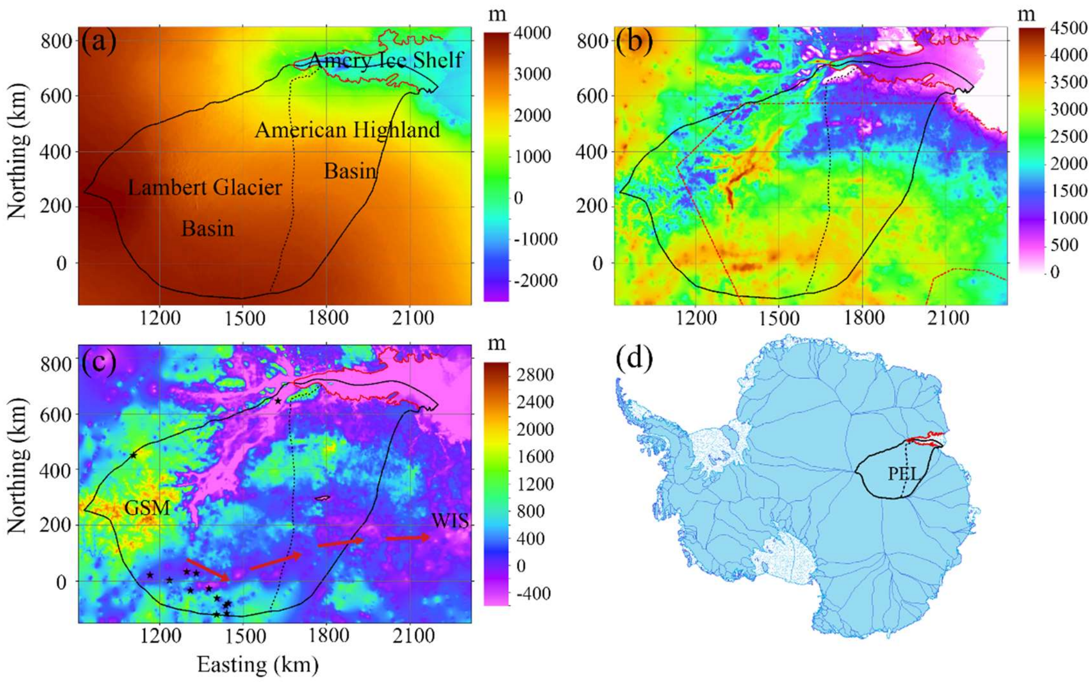
Reply: Disagreed. We think "we make inferences on …" is correct usage here.

Section 2

L103 and fig2 –rephrase, and draw the whole shelf/gl so that the reader can easily tell what is meant by 'half'. Related to this, in 3.3.3 explicitly state the boundary condition at this segment of lateral boundary (I assume it is the same as the other boundaries) and give a justification.

Reply: We assume you mean to draw the whole shelf/gl in Fig. 1 rather than Fig. 2, because Fig. 1 is included in our sentence "It consists of two drainage basins: the Lambert Glacier Basin, the American Highland Basin, along with about half of Amery Ice Shelf (Fig. 1)."

In the updated Fig. 1, we draw the whole grounding line of Amery ice shelf, we added "about" in the description along with about half of Amery.



The updated Fig. 1 in the revision. The domain topography and location with domain boundary overlain. (a) surface elevation; (b) ice thickness; (c) bed elevation; (d) the location of our domain in Antarctica. The solid black curve is the outline of the study domain, including the central streamline of Amery ice shelf and the boundary of inland sub-basins based on drainage-basin boundaries defined from satellite ice sheet surface elevation and velocities (Mouginot et al., 2017; Rignot et al., 2019). The solid red curve is the grounding line of Amery ice shelf (Morlighem et al., 2020). The dotted black curve is the dividing line between Lambert Glacier Basin and the American Highland Basin. The dotted red curve in (b) is the boundary of ice thickness data from Cui et al. (2020a). The black stars in (c) denote the locations of observed subglacial lakes (Wright and Siegert, 2012; Cui et al., 2021), the region within the black line at (1800E, 300N) is potentially the second largest subglacial lake in Antarctica. The red arrows in (c) indicate the routing through the deep subglacial canyon system from GSM to WIS.

L:156 'Inverse method'- no such thing. You are solving an inverse (that is, ill-posed) problem, using (most likely) some sort of gradient based optimization method. You are also not estimating ice flow velocity and stress, but inferring the basal friction such that the model velocity best fits observations.

Reply: We change the sentence in L156 to "We solve an inverse problem by a full-Stokes model, implemented in Elmer/Ice, to infer the basal friction coefficient such that the model velocity best fits observations (Gagliardini et al., 2013). Using the best-fit basal friction coefficient, we obtain the ice flow velocity, stress, and basal friction heat."

We also change "inverse method" elsewhere to "inverse problem" or "inverse model".

L158-163 – Some rewording is needed here. You don't describe the procedure that you hint at for some time, so provide a summary here ('we will describe each model component in sections X and Y, then the coupling in Z')

Reply: Thanks for your comments. We adjust the structure and provide a summary here. "We will describe the forward model in Section 3.1 and the inverse model in Section 3.2, then the coupling in Section 3.3."

L223 In general, this section need to be cleared up, how for example does 'water input supply a large freezing rate'.

Reply: Most of this section has been removed, since another referee said we could just cite this paper rather than copy all these sections, and we just discuss the differences in setup from Wolovick et al. (2021).

L219: no need to say 'taking six GHF datasets…' or at least rephrase to be clear that you only use one at a time.

Reply: This sentence is removed.

L329; Eq 15 is not the Weertman law, it is a linear viscous law which works satisfactorily in inverse problems (because you are really finding Tb, not C) but not in general.

Reply: We change "Weertman law" to "a linear sliding law".

L347 Use subscripts consistently

Reply: we change u^{obs} to u_{obs}.

L363 (and elsewhere) the conductive heat flux Fc = -k dT/dz is positive (upward) when the bed is warmer than the ice above, so should you not have + k dT/dz (i.e – Fc) if the bed "loses heat from upward heat conduction" . What about the case where dT/dz is negative (pressure melting point reached above the ice bed). Does that simply never happen?

Reply: Yes, we note there is a sign typo in this equation, it should be as below

$$M = \frac{G + \vec{u}_b \tau_b + k(T)\frac{dT}{dz}}{\rho_i L}$$

where the term $k(T)\frac{dT}{dz}$ is negative, representing heat loss of basal ice by upward englacial heat conduction.

We add this paragraph in the revision: "In the case that the modelled basal ice temperature reaches pressure melting point, T_m, a temperate basal ice layer is permitted in our model. The model works with englacial melting and a temperate ice layer. We do not make assumptions about liquid pore water content. We use a weak-form solution instead of a strict limit. The temperature is allowed to exceed the melting point, but temperature rise is limited by the latent heat absorbed by englacial melting. So, the melt rate rises exponentially as temperature passes the melting point, and the pre-factor for the melt rate comes from the strain heating. "

L400. This procedure seems important but is glossed over. If Bnew != Bold, then why does the modelled surface velocity not change?

Reply: We note in the text "the difference of simulated and observed surface velocity is unchanged in the whole region except for some parts of the inland boundary." We note in the conclusions "We also improve the basal friction calculation to include information on the basal ice temperature relative to its pressure melting point. This procedure results in removal of unrealistic noise manifested as local spikes in modelled basal friction heat. "

We add a figure as below to compare modelled basal friction heat before and after this improvement of β.

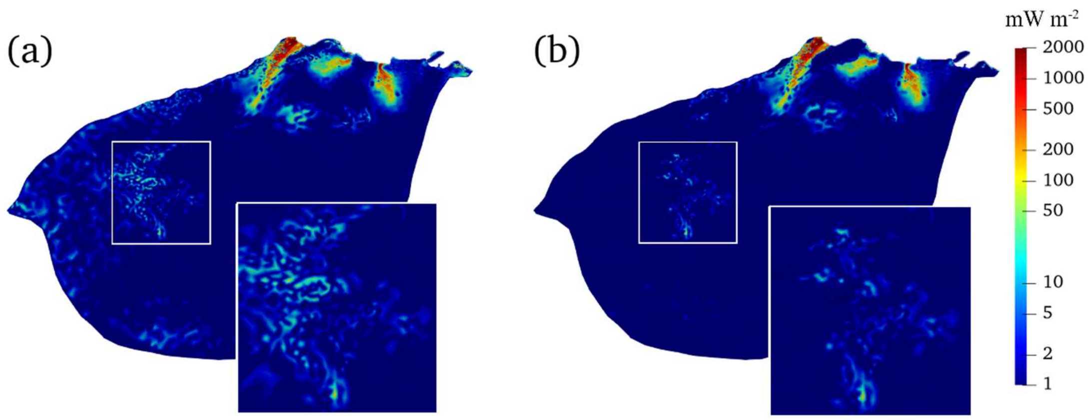


Figure: Comparison of modelled basal friction heat with basal friction coefficient β_{old} (a) and β_{new} with α=1 (b). The white square is enlarged.

L450 and fig 6. Why is the heat flux negative? Especially since you talk about magnitudes in the text.

Reply: In the revision, we change "heat conduction" in figure caption of Figure 6 to "modelled heat change of basal ice by upward englacial heat conduction", and add more sentences "The negative sign means that the upward englacial heat conduction causes heat loss from the basal ice as defined by the color bar with cooler colors representing more intense heat loss by conduction."

We also change the text correspondingly.

L460 and fig 7 – it is difficult to tell the difference between these. Would it help to show differences relative to (a) Martos? That said, you don't seem to depend much on these figures so are they really needed?

Reply: Agreed. We can see no significant difference across these 6 experiments, We add a new plot (the new Fig. 4) in the revision showing the modeled basal friction heat before and after the improvement of β using Martos GHF. So we do not need the old plot Fig. 7 in the revision. We change in the revision "There is no significant difference in modelled basal friction heat across these 6 experiments. We only show the modelled basal friction in experiment using Martos et al. (2017) GHF (Fig. 4b)."

L487. It is not quite accurate to say that 'The Li experiment gives the best fit' (how is the fit quantified?). I suggest rephrasing along the lines of the following sentence which sums up the results more accurately, i.e it is only the Li experiment that results in a warm base that covers all observed lakes.

Reply: We rephrased these sentences as you suggested "The modelled warm base in the experiment using Li et al. (2021) GHF covers all the observed subglacial lakes in the domain, including the recently discovered second-largest subglacial lake in Antarctica (Cui et al., 2020b). The warm base in the experiment using Martos et al. (2017) GHF covers the second most observed subglacial lakes, and the experiment using An et al. (2015) GHF the third".

L506-507 (and elsewhere): The datasets/fields should be referenced correctly '(Li et al 2021)', rather than just 'Li'. I also would prefer to see you write 'our experiment using the Li et al 2021 GHF' rather than 'the Li experiment', but I don't think there is any real danger of the reader being misled by that.

Reply: We change "Li" to 'Li et al. (2021)', and similar for other GHF datasets. We also change 'the Li experiment' to 'experiment using the Li et al. (2021) GHF', and similar change for other GHF datasets all through the text.