Referee's comments are in blue, our reply in black, quotes in the revised manuscript in purple.

Kang et al. evaluated the basal thermal conditions of the Lamber-Amery system by using a combined model of a forward model and an inverse model. Results from six experiments based on different geothermal heat flux (GHF) products indicated different distributions of basal temperature and modelled basal melting. By comparing the modelled warm-based region and basal melting rates with locations of subglacial lakes, this study found that two most-recent GHF products based on aerial geomagnetic observations provided best constrain as the basal thermal conditions. Overall the manuscript is generally clearly written. However, the structure needs further modification and some of the description and figures need more improvements.

Here are some general comments:

The finding about consistency between the high basal friction heating and the fast-flowing regions can be easily seen from the way how you calculate the friction heating (Q=tao*velocity), which is less innovative as one of findings in a high-quality peer-reviewed paper.

Reply: Agreed. We remove the sentences in the abstract and conclusion talking about the consistency between the high basal friction heating and the fast-flowing regions.

The section of 3.1.1, 3.1.2, 3.1.3 is nearly same with Wolovick et al. (2021). The authors could just cite this paper rather than copy all these sections. Just make it clear about the different setup you used from Wolovick et al. (2021).

Reply: We removed most words and the separate Section of 3.1.1, 3.1.2, 3.1.3, and pointed the different setup we used from Wolovick et al. (2021), which is mainly about how we used a merged surface flow direction field, and how we use basal sliding ratio computed by the full-Stokes inverse model to constrain rheology and shape function model in the forward model.

The structure of the paper is a little bit confusing. I suggest moving Sec. 4.2 to Sec. 3.2. Sec. 4.1 Experiment design could fit into end of Sec. 3. Leave Sec. 4.3 as a separate section Sec. 4.

Reply: Thank you for the suggestions. We modified the structure of the paper as you suggested. Due to the changes in text, we also reordered the number of equations.

About the improvement of basal friction coefficient, is it original from this study? If yes, I suggest you to mention it in your conclusion section. Besides, I did not see any evaluation about this improvement. Comparison of the difference of simulated and observed surface velocity before and after this improvement is necessary here.

Reply: Yes, the improvement of basal friction coefficient is original from this study. We mention it in our conclusion section of the revision. We also improve the basal friction calculation to include information on the basal ice temperature relative to its pressure melting point. This procedure results in removal of unrealistic noise manifested as local spikes in modelled basal friction heat.

The goal of this improvement is to remove unrealistic noise manifested as local spikes in modelled basal friction heat. We show two comparison plots below, one shows the comparison of modelled basal friction heat before and after this improvement, the other shows the difference of simulated and observed surface velocity before and after this improvement.

We can see that the unrealistic noise is much less after this improvement, and the difference of simulated and observed surface velocity is unchanged in the region except for some parts of the inland boundary.

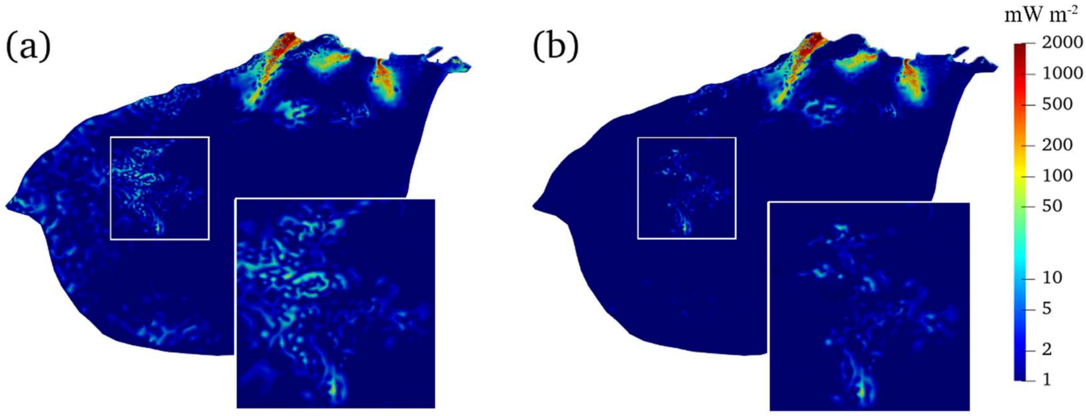
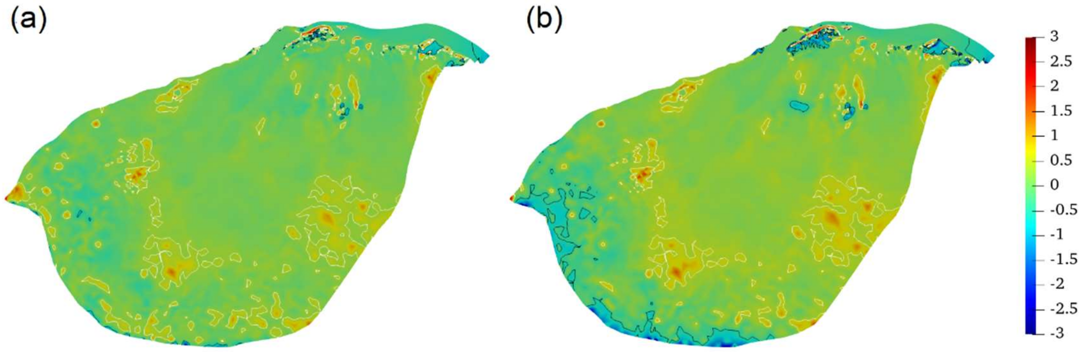


Figure: Comparison of modelled basal friction heat with basal friction coefficient β_{old} (a); and β_{new} with α=1 (b). The white square is enlarged.



This figure (not in the manuscript) shows the difference between simulated and observed surface velocity plotted as $\log_{10}$(modeled/observed) using different basal friction coefficients β_{old} (a); and β_{new} with α=1 (b). The white lines in represent contours of 0.5 (a ratio of modeled/observed of about 3) and the black lines represent contours of -0.5 (a ratio of about 1/3).

When you talk about the effects of different GHFs on the modelled basal melting, you ignored that fact that different GHFs only affect the modelled basal melting in low-flowing regions even if those six GHFs show different distribution in the fast-flowing region. It further confirmed that friction heating dominated the basal melting for fast-flowing region while the GHF dominated the basal melting in slow-flowing region.

Reply: We do not fully agree with your opinion that "different GHFs only affect the modelled basal melting in low-flowing regions". GHFs not only affect the extent of basal melting but also affect the magnitude of basal melting rate. Although there is basal melt in fast-flowing region using different GHFs, the magnitudes of basal melt rates are different. For instance, use of Purucker GHF which is lower than other GHFs in the fast flow region produces smaller basal melt rate in the fast-flowing region. We added: The fast-flowing region has smaller modelled basal friction coefficients, and faster basal velocities, but there are large differences in basal melting rates between the 6 GHF datasets.

I don't think the Abstract and Conclusions highlight all of the valuable findings in this study. I suggest a serious revision on it.

Reply: We revised the Abstract and Conclusions in the revision.

Abstract:

Basal thermal conditions play an important role in ice sheet dynamics, and they are sensitive to geothermal heat flux (GHF). Here we estimate the basal thermal conditions, including basal temperature, basal melt rate, and friction heat underneath the Lambert-Amery glacier system in east Antarctica, using a combination of a forward model and an inversion from a 3D ice flow model. We assess the sensitivity and uncertainty of basal thermal conditions using six different GHFs. We evaluate the modelled results using all observed subglacial lakes. The different GHFs lead to large differences in simulated spatial patterns of temperate basal conditions. The two recent GHF fields inverted from aerial geomagnetic observations have the highest GHF, produce the largest warm-based area, and match the observed distribution of subglacial lakes better than the other GHFs. The modelled basal melt rate reaches ten to hundreds of mm per year locally in Lambert, Lepekhin and Kronshtadtskiy glaciers feeding the Amery ice shelf, and ranges from 0-5 mm yr^{-1} on the temperate base of the vast inland region.

Conclusions

In this paper, we estimate the basal thermal conditions of the Lambert-Amery system by coupling a forward model and an inverse model, based on six different GHF datasets. We analyze the contribution of GHF, heat conduction, and basal friction to the modelled basal melt rate. We verify the result using the locations of all known subglacial lakes, and evaluate the reliability of six GHF datasets in our study domain.

Our approach is distinct from that used to find GHF fields employed by Wolovick et al. (2021a), in particular the use of a full Stokes model allows the method to be extended to fast flowing ice stream and ice shelf domains where neither the shallow ice or shallow shelf-approximations are valid. We also improve the basal friction calculation to include information on the basal ice temperature relative to its pressure melting point. This procedure results in removal of unrealistic noise manifested as local spikes in modelled basal friction heat.

We find significant differences in the spatial extent of temperate ice in the slow flowing areas among the six experiments due to large variability in GHF. The experiments using Li et al. (2021) and the Martos et al. (2017) GHF yield the largest area with basal melting, and match the subglacial lake locations best. In contrast, the experiments using Purucker (2013) GHF gives the least area with basal melting and the worst match with subglacial lakes locations. We suggest GHF datasets from Li et al. (2021) and Martos et al. (2017) as the most suitable choice for this study region. We cannot make our own GHF map from our analysis since while we can pick the GHF where Li and Martos geothermal heat flow maps are consistent and both agree with the observations, we do not know which (if either) are correct where the Li and Martos GHF datasets disagree and there are no observations. In order to make this determination we would need additional observational constraints on the basal thermal state, such as measured basal temperatures from deep ice cores, or observed refreeze-on, but neither are available in the region.

The fast-flowing region has smaller modelled basal friction coefficients, and faster basal velocities, but there are large differences in basal melting rates between the 6 GHF datasets. The fast-flowing tributaries have frictional heating in the range of 50-2000 mW m^{-2}. In the vast inland areas, our experiments generally yield high upward heat conduction in the range of 45-60 mW m^{-2} which means that GHF dominates the heat content of the basal ice in the slow flow regions. The modelled basal melt rate reaches 50-500 mm yr^{-1} locally in three very fast flow tributaries (Lambert, Lepekhin and Kronshtadtskiy glaciers) feeding the Amery ice shelf, and is in the range of 0-5 mm yr^{-1} in the inland region.

Several places across the text are lack of citations or need more relevant literature. Some of the figures are not cited accordingly in the text. See the details below.

Specific Comments:

L37: "evidence of extensive subglacial rifts and lakes" citation please.

Reply: We add references.

L77: "for ice temperature" → "ice temperature simulation".

Reply: done.

L83-85: Unfinished sentence I guess. "inferred ice and basal temperature"? Or I misunderstood your meaning here.

Reply: We change this sentence "Large scale studies on the dependence on GHF of the Greenland (Rezvanbehbahani et al., 2019) and Antarctica ice sheet (Pattyn, 2010) have inferred ice and basal temperatures" to "Glaciologists have combined ice sheet models with measurements of vertical temperature or thawed basal state to constrain GHF of the ice sheets (e.g. Pattyn, 2010; Rezvanbehbahani et al., 2019)".

L101: "in" → "part of"

Reply: Done.

L104: How did you choose the central streamline here? Where are those datasets (basin boundary, ice front) from? Please add citations.

Reply: We made it ourselves. The central streamline was chosen by selecting a point at

the confluence of Lambert Glacier and Lepekhin Glaicer and then advecting that point downstream to the ice front using the observed velocity field.
L115-117: citation of the grounding line dataset and the subglacial lakes.
Reply: We add the citations. "The red curve is part of the grounding line of Amery ice shelf (Morlighem et al., 2020) … The black stars in (c) denote the locations of observed subglacial lakes (Wright and Siegert, 2012; Cui et al., 2021)"
L123-124: It's not clear to me how and where these two datasets are combined. You should make it clear in Fig. 1.
Reply: In Fig. 1, we add a dotted red curve in plot (b) showing the boundary of ice thickness data from Cui et al. (2020a). We use the data from Cui et al. (2020a) inside this boundary and BedMachine data outside this boundary.
L156: This is your first time to mention inverse method and Elmer/Ice. Please add citations.
Reply: Done.
L319: In the boundary condition section (Sec. 3.2.2), you did not mention the constrain for the surface mass balance and basal mass balance for the floating part. Please make it clear here.
Reply: In Elmer/Ice model, we do diagnostic simulation, i.e., we perform a stress-balance snapshot. Therefore, we do not need to prescribe surface mass balance or basal mass balance in the boundary conditions for the ice sheet including the ice shelf. We add the explanation in section 3.2.2.
L362: This equation is not clearly explained. What is each component in the numerator? Please also add citations for this equation.
Reply: There was a typo in Eq (19). We corrected it in the revision.

$$M = \frac{G+\vec{u}_b\tau_b+k(T)\frac{dT}{dz}}{\rho_i L}$$

where M is the basal melt rate, G is GHF, $\vec{u}_b\tau_b$ is the basal friction heat, $-k(T)\frac{dT}{dz}$ is the upward heat conduction, n is the outward unit vector at the ice bottom, ρ_i is the ice density, and L is latent heat of ice melt. We add the reference for this equation:
Greve R, Blatter H, Dynamics of Ice Sheets and Glaciers, Springer, 2009.
L368: The experiment design is quite similar to the multi-cycle spin-up used in Zhao et al. (2018). If yes, please cite the paper here.
Reply: It is similar. We cite the paper Zhao et al. (2018).
L395: citation for the statement "Basal friction in reality depends on basal temperature"
Reply: We add the reference:
Greve R, Blatter H, Dynamics of Ice Sheets and Glaciers, Springer, 2009.
L415: delate "the" after "the modelled".
Reply: Done.
L416: Do you mean test with different GHFs gave you similar modelled surface velocity? If yes, the statement you made here is not accurate. The only thing you can say is that

the inverse method is not sensitive to the choice of GHF product as the boundary condition, which could be one of your findings here.

Reply: The inverse method is designed to minimize the misfit between modelled and observed surface velocity. Therefore, it is not surprising that the modelled surface velocities are similar for the different GHFs. This is not a finding. It is what one expected.

We change "In the inverse method, the modeled surface velocity matches best to the observed surface velocity. Therefore, we get very similar distributions of modeled velocity field using different GHFs" to "In the inverse method, the misfit between the modeled and the observed surface velocity is minimized. Therefore, we get very similar distributions of modeled surface velocity field using different GHFs.".

L427: 500 m/yr. Do you mean the velocity near the GL? If yes, make it clear.

Reply: Yes, done.

L432: The cyan color is not clear to me. Suggest to change a different color.

Reply: We update Fig. 4 in the revision. We use white solid lines in (a), (b), and (d) to plot speed contours of 50, 100 and 200 m yr^{-1}.

L433-434: Why do you chose the contour of 0.5 and -0.5 here? What's the meaning behind those two contours. Please explain.

Reply: We do not think this subplot is helpful, so we remove it in the revision, just using it reply to your earlier general comment above.

The values are arbitrary and simply show the ranges of the velocity differences. The contour 0.5 means $\frac{\text{modelled velocity}}{\text{observed velocity}} = 10^{1/2} \approx 3.1$, and the contour -0.5 means $\frac{\text{modelled velocity}}{\text{observed velocity}} = 10^{-1/2} \approx \frac{1}{3}$. We use ratio of 1/3~3 times to compare the difference of modelled and observed velocity. Modelled velocity in most region is in this range.

L339-440: But for the fast-flowing region, we did not see any significant differences. You should make it clear when you talk about the different distribution of warm base.

Reply: We assume you mean L439-440. We modify it to "The modelled ice bottom of fast-flowing region are all warm based (basal temperature reaching the pressure melting point). However, there are significant differences in the modelled distribution of warm base in the slow-flowing region using different GHFs."

L441: "In the Li experiment", please cite the figure here. "high" → "highest"

Reply: Done.

L442: "the basal temperature over most of the domain reaches the melting point", you should add "except for the southern part of domain"

Reply: Done.

L447: citation for "subglacial mountains"

Reply: Subglacial mountains are shown in Fig. 1c. so we refer to Fig. 1c here.

L455: "heat conduction" → "basal heat conduction". Please add the velocity contour in Fig. 6. About the "fast-flowing tributaries", you didn't define it in Fig. 4a. Do you mean region with velocity higher than 50 m/yr?

Reply: To be more clear, we changed "heat conduction" to "modelled heat change of basal ice by upward englacial heat conduction".

The fast-flowing tributaries, we mean the region with velocity higher than 30 m/yr. We add velocity contours of 30, 50, 100, 200 m/yr in Fig. 6 (which is Fig. 8 in the revision).

L456: "0-30" → "30"

Reply: Done.

L457-459: Why do you think Purucker shows lower values here? Please explain.

Reply: This sentence is only for vast inland areas (slow-flowing region). We can tell that from the colorbar in Fig. 6. Purucker (Fig. 6e) has lighter color than other subplots.

L460: From Fig. 7, we can tell no significant difference across these 6 experiments. It's better to make a statement here.

Reply: We add a statement here "There is no significant difference in modelled basal friction heat across these 6 experiments."

L463: when you say reach 2000 mW m^{-2} at the GL, do you mean all these three glaciers? Or just Lambert?

Reply: We change it to "The three fast-flowing tributaries have friction heat amounting to more than 50 mW m^{-2}, with the Lambert and Kronshtadtskiy glaciers having 2000 mW m^{-2} at the grounding line."

L478: there are two Fig. 8 here.

Reply: we remove one.

L505: I think GHF distribution largely govern basal thermal conditions for the slow-flowing region. Add citations for "Many previous studies"

Reply: Done. We add citations Larour et al., 2012; Pattyn, 2010; Pittard et al.,2016; Van Liefferinge and Pattyn, 2013; Van Liefferinge et al. 2018.

L511-L515: Too long sentence. Please split it.

Reply: we change this sentence to "However, it should be noted that observations of subglacial lakes are a one-sided constraint. A model result that misses the observed lakes is clearly too cold at that location. But if the model result shows basal melt at a place with no observed lakes, it is not clear whether this is because the model is too warm, or if the subglacial water exists in a form other than in ponded lakes.".

L513: Don't understand what you mean here by "puts warm-based conditions outside of the locations of the observed lakes"

Reply: We mean "if the model result shows basal melt at a place with no observed lakes", see the above reply.

L514: delete "if"

Reply: see the above reply.

L517: I don't think you use the same inversion method by Wolovick. Do I misunderstand anything here?

Reply: That is correct, we do not use the same inversion method as Wolovick et al (2021). That paper adjusted GHF and surface accumulation rate to fit observations of subglacial lakes, basal freeze-on, and internal layers. We only use the forward model described in that paper for our thermal and hydrology model. The inverse model used here, by contrast, is a classical ice dynamic inversion that adjusts basal friction to match surface velocity.

We change this sentence “Our methodology builds on the earlier inversion method employed by Wolovick et al. (2021)” to “Our approach is distinct from that used to find GHF fields employed by Wolovick et al. (2021a), in particular the use of a full Stokes model allows the method to be extended to fast flowing ice stream and ice shelf domains where neither the shallow ice nor shallow shelf-approximations are valid.”

L520: What is “ice bed”?

Reply: we change it to “ice bottom”.

L525: So what? What is the advantage behind it? This could be a highlight of your study.

Reply: We mention this in the conclusions: We also improve the basal friction calculation to include information on the basal ice temperature relative to its pressure melting point. This procedure results in removal of unrealistic noise manifested as local spikes in modelled basal friction heat.

The goal of this improvement of β is to reduce the local spikes in modelled friction heat. The modelled surface velocity after the improvement of β is unchanged in the region except for some parts of the inland boundary.

L542: what do you mean by “ice sheet connected to the ice shelf”? “frictional heating means”? This sentence is not clear to me.

Reply: Sorry that we did not express clearly. It means “grounded ice sheet near the ice shelf”. We change it to “Most GHF distributions (except Martos et al., 2017 and Li et al., 2021) in the grounded ice sheet near the ice shelf are homogeneous, but frictional heating in the fast-flowing ice is more than 10 times higher than in the slow-flowing ice.”

L555: delete “,”

Reply: done.

L573: “in area” →“in slow flowing area

Reply: done.