*I have responded in italics to the comments that I feel may not have been addressed as intended*

- For the most part, though some of the derivation needs a bit of clarification; likely the results hold, but as a reader I struggled to follow some of the steps. Sometimes the assumptions previously stated should be reiterated (especially in the discussion) to ensure the readers understand the limitations of this model.

We included the complete assumptions for the theoretical equation in addition to the quasi-steady state approximation (Line 152-155).

The material of permafrost and talik is assumed to be fully saturated with ice and water, respectively. Also, the thermal constants (thermal conductivity, latent heat, and thawing temperature) are constant and isotropic, and the change in volume of water on thawing and freezing is negligible. Under such assumptions...

*Are there any other assumption – i.e. equilibrium conditions, constraints on processes contributing to energy balance..*

- I am unsure of the comparison with field data - I think either this section should be expanded to include more sites, removed (which I am sure the authors agree would detract from the merit of this contribution), or perhaps re-phrased as an example application of this new method and not a test of the method proving its efficacy.

It is hard to obtain the talik depth measurements under an isolated lake in a continuous permafrost as stated above.

*I would still recommend re-framing this contribution then as a numerical model, and using this lake as an example as opposed to a test of the model efficacy.*

- L 23 ... the Euler equation and the calculus of variations

The original manuscript focused on the mathematical technique, which appears as “Euler equation in the calculus of variation”. We propose more complete explanation with the physical context beyond the Newtonian mechanics, which hopefully helps readers to understand the background of this idea in some extent. “Euler–Lagrange equation” replaces “Euler equation in the calculus of variation” in the revised manuscript.

We added the following paragraph at the beginning of Chapter 2, Theory (Line 132 – 147).

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The approach used in this study is based on Lagrangian mechanics, which generalizes the classical Newtonian mechanics, using the stationary action principle (the principle of least action). The action is defined as the integral of the Lagrangian, which consists of kinetic and potential energy of the system. In this application, the Lagrangian simply becomes the potential energy due to absence of kinetic energy. The variational principle that is the main tool in Lagrangian mechanics can *be used to* derive the equations in Newtonian mechanics. One of the related research topics using the variational principle to fluid mechanics is phase boundary propagation, which can be analyzed by the phase field model or diffusion-interface model (Cassel, 2013). This model explains the diffuse phase boundary without surface tension that appears in Newtonian interfacial physics between a liquid and a gas. According to the second law of thermodynamics, the free energy of the system must decrease monotonically to ensure a non-negative entropy production (Singer-Loginova and Singer, 2008). This requires that the time rate of change of the phase boundary be expressed by the functional derivative of the free energy function~~al~~, which corresponds to the talik total energy flux in relation to permafrost thaw. This study directly and analytically solves the Euler-Lagrange equation based on the stationary action principle rather than the entropy functional used in the phase field method.

*I do not understand this description, and therefore cannot evaluate its validity – additional references would be helpful, but overall I think an expert in Lagrangian mechnanics is needed to understand if this method is appropriate in this situation.*

*“* the talik total energy flux in relation to permafrost thaw” *– what does this mean? The total energy flux into the talik?*

Also, Equation (15) uses the method of Lagrange multipliers which is a common approach in machine learning field, lately. We hope the name of the method helps for readers to understand the physical interpretation.

*Definitely helpful, but some physical interpretation needed. The application of the model to the lake in question should include some of this translation of the theory to measurements.*

- l 341 is the assumption that the radial thermal gradient is zero accurate? Other publications report much more rapid horizontal than vertical thaw (though my focus is discontinuous PF) see McClymont et al. Devoie et al. work at Scotty Creek. Please cite something or report thermal gradients to support this.

Thank you. We cited McClymont et al. (2013) and Devoie et al. (2021) to support the approximation that the inter-seasonal average of the horizontal thermal gradient is negligible (Line 376-377)

*This is not what was expected – these publications show that lateral thaw is *FASTER* not slower, and do not comment on the lateral gradient. This also belongs in the model assumptions list – it is critical to the representation of the system, and therefore should be clearly listed as a limitation of the model.*

- l 371 the preceding discussion all hinges on the zero lateral gradient assumption - please highlight this otherwise it seems unlikely

Thank you. We revised the assumption statement as “... assuming all other properties and horizontal thermal gradient variation are equal...” on Line 400-403.

*Are equal to what? The horizontal thermal gradient = 0, the other properties are fixed?*

- l 535 what about anisotropic thermal properties? Maybe also discuss these as well?

Sorry. We could not catch it. Thermal properties (e.g. thermal conductivity, latent heat, and thawing temperature) should be isotropic, constant, and uniform. However, the thermal field is anisotropic (e.g. vertical temperature gradient).

*Yes, this is your assumption, but is it reflected in reality? How may this be a source of error in your model’s prediction?*