Our responses in orange color are inserted between the comments and notes.

Dear authors,

The reviewers have now provided their reports on your revised version and have acknowledged that the manuscript has significantly improved. One reviewer mentions that the description of the model approach is now sufficiently detailed, whereas the other mentions some minor concerns regarding some of your responses/changes, which may warrant some modifications. They are especially addressing the uncertainties involved with too simplistic model assumptions, as well as the framing of the study around the question whether one field site is sufficient to "test" a model approach or whether this test case should be seen as an example of the application of the model.

Please address these comments as usual point-by-point. We are very much looking forward to your response and the revised manuscript.

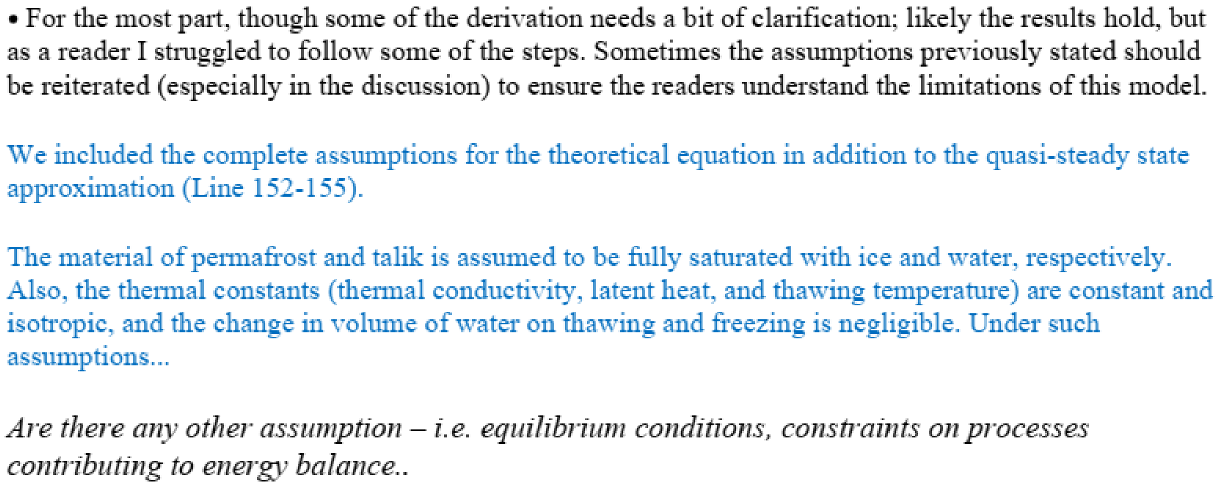
kind regards,

Christian Hauck

Editor

Thank you for examining the revised draft and giving us another opportunity to improve the manuscript. Our responses to your suggestions are included below between the reviewer’s comments denoted by orange color this time.

Reviewer’s comments:



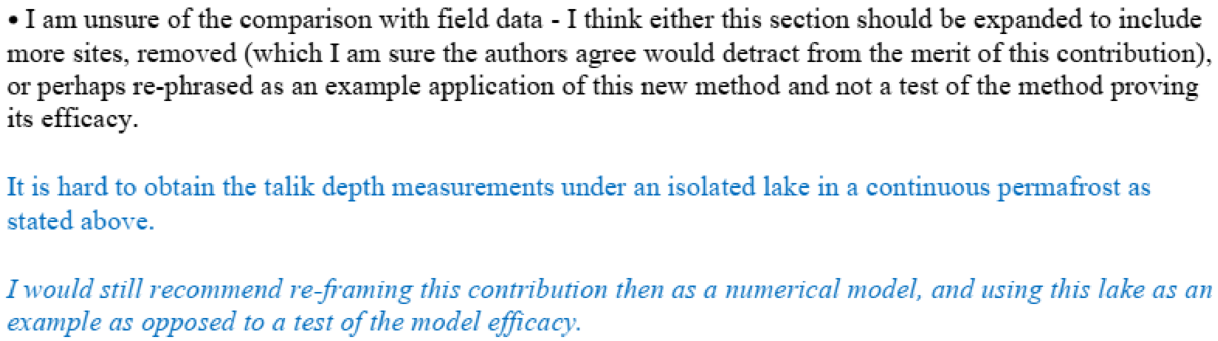
- For the most part, though some of the derivation needs a bit of clarification; likely the results hold, but as a reader I struggled to follow some of the steps. Sometimes the assumptions previously stated should be reiterated (especially in the discussion) to ensure the readers understand the limitations of this model.

We included the complete assumptions for the theoretical equation in addition to the quasi-steady state approximation (Line 152-155).

The material of permafrost and talik is assumed to be fully saturated with ice and water, respectively. Also, the thermal constants (thermal conductivity, latent heat, and thawing temperature) are constant and isotropic, and the change in volume of water on thawing and freezing is negligible. Under such assumptions...

Are there any other assumption – i.e. equilibrium conditions, constraints on processes contributing to energy balance..

The quasi-steady state is equivalent to the (quasi) equilibrium state in this case. Note that Equation 1 is a dynamic equation (neither steady nor equilibrium) as time is involved in it. The listed assumptions covered all the constraints on the processes considered in permafrost thaw.



- I am unsure of the comparison with field data - I think either this section should be expanded to include more sites, removed (which I am sure the authors agree would detract from the merit of this contribution), or perhaps re-phrased as an example application of this new method and not a test of the method proving its efficacy.

It is hard to obtain the talik depth measurements under an isolated lake in a continuous permafrost as stated above.

I would still recommend re-framing this contribution then as a numerical model, and using this lake as an example as opposed to a test of the model efficacy.

Although we had not intended to “test” the theory by using single lake data, we rephrased several parts in the manuscript to emphasize that this is an example application of this new method.

Line 29

> The model predicting ellipsoidal talik geometry was compared to talik thickness observations using transient electromagnetic (TEM) soundings in Peatball Lake on the Arctic Coastal Plain (ACP) of northern Alaska.

Line 335-340:

> Despite irregularity due to the complex lake formation history, the overall lake talik geometry may be approximated by a semi-ellipsoidal shape as indicated by the very good fit of the elliptic model to the TEM measured talik thicknesses (see Figure 4 with overall RMSD = 5.94 m, 6.7 % of the maximum talik depth). The idealized, thermally optimum model geometry can partition talik irregularity associated with multi-generation lakes such as Peatball Lake.

Additionally, it may be pointed out that other numerical models suggested an elliptic/ellipsoidal talik geometry. We added one sentence at the end of derivation on Line 252-255:

> This result is consistent with the existing numerical thermal models (Schwamborn et al., 2000; Ling and Zhang, 2003b; Plug and West, 2009; Kessler et al., 2012) which predicted nearly elliptic talik cross sections under thaw lakes in continuous permafrost.

For convenience, four visualized model outputs are shown below:

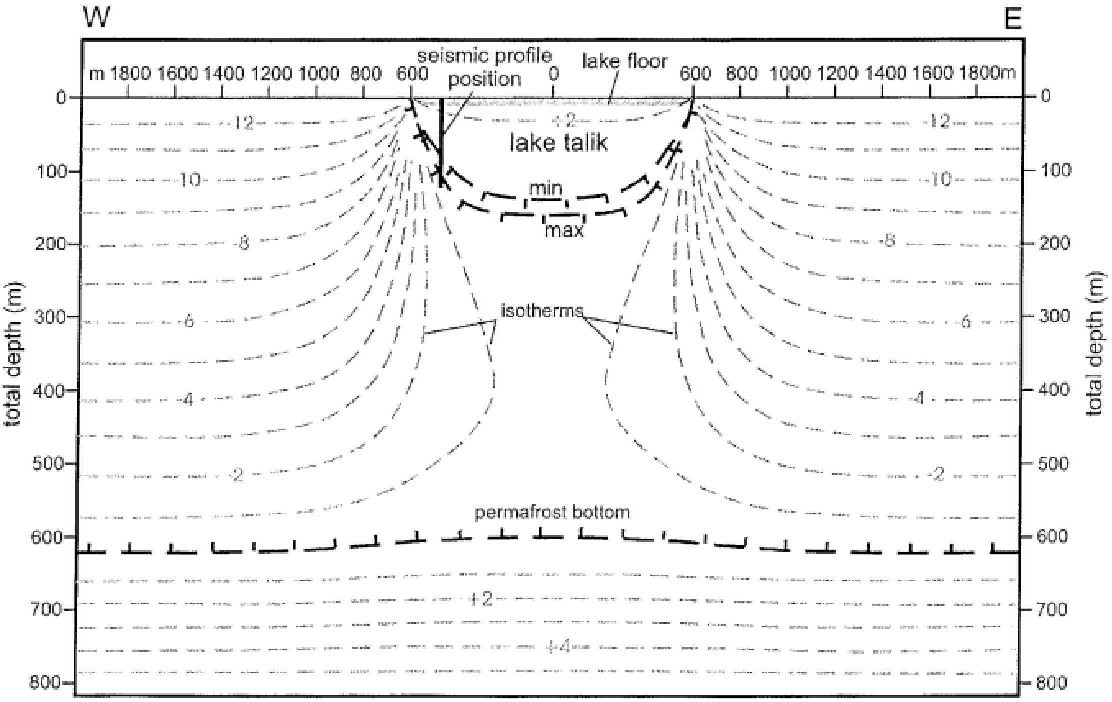


Figure 8 in Schwamborn et al. (2000)

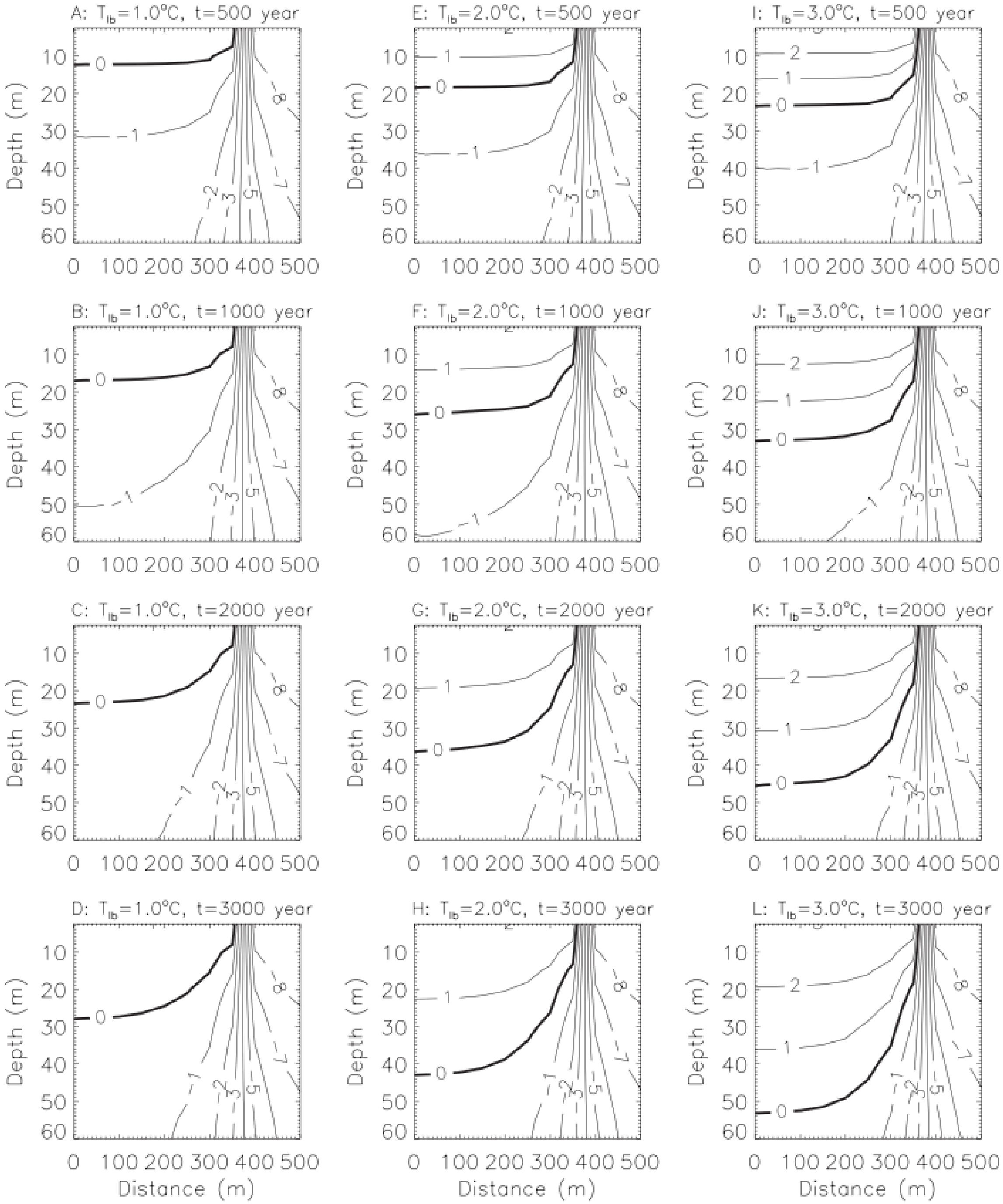


Figure 7. Simulated ground thermal regimes and talik thicknesses under thaw lakes at 500, 1000, 2000, and 3000 years for simulation cases C9, C10, and C11.

Figure 7 from Ling and Zhang (2003b)

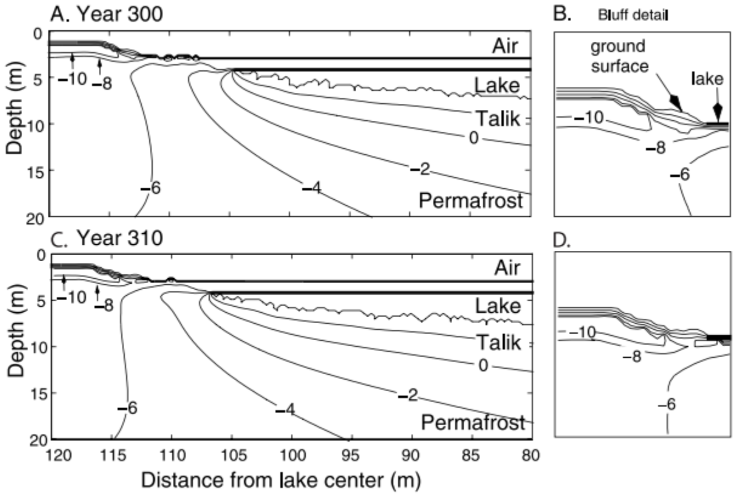


Figure 2. Thermal and morphological profiles of a simulated lake after (a and b) 300 years and (c and d) 310 years in one realization using NSP model parameters. Time of year is the end of winter; lake ice lies between indicated air and lake. Over the 10 year period shown, the basin (defined by the top of the bluff) has expanded 2.4 m compared to ≈0.5 m for the lake, the latter less because of sediment deposited at the lake margin prior to year 300 (which has partly diffused into the basin by year 310) (Figures 2b and 2d). Water depth and talik thickness at the lake center remained constant because basin subsidence approximately balanced sediment deposition.

Figure 2 in Plug and West, 2009.

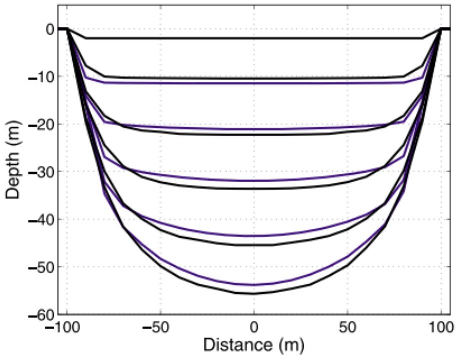


Figure 4. Downward progression of the talik, shown at years 0, 50, 200, 500, 1000, and 1500, for the thermal model used here (blue) and a 3-D finite difference solution of the heat equation (black).

Figure 4 from Kessler et al (2012)

- L 23 ... the Euler equation and the calculus of variations

The original manuscript focused on the mathematical technique, which appears as “Euler equation in the calculus of variation”. We propose more complete explanation with the physical context beyond the Newtonian mechanics, which hopefully helps readers to understand the background of this idea in some extent. “Euler–Lagrange equation” replaces “Euler equation in the calculus of variation” in the revised manuscript.

We added the following paragraph at the beginning of Chapter 2, Theory (Line 132 – 147).

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The approach used in this study is based on Lagrangian mechanics, which generalizes the classical Newtonian mechanics, using the stationary action principle (the principle of least action). The action is defined as the integral of the Lagrangian, which consists of kinetic and potential energy of the system. In this application, the Lagrangian simply becomes the potential energy due to absence of kinetic energy. The variational principle that is the main tool in Lagrangian mechanics can *be used to* derive the equations in Newtonian mechanics. One of the related research topics using the variational principle to fluid mechanics is phase boundary propagation, which can be analyzed by the phase field model or diffusion-interface model (Cassel, 2013). This model explains the diffuse phase boundary without surface tension that appears in Newtonian interfacial physics between a liquid and a gas. According to the second law of thermodynamics, the free energy of the system must decrease monotonically to ensure a non-negative entropy production (Singer-Loginova and Singer, 2008). This requires that the time rate of change of the phase boundary be expressed by the functional derivative of the free energy ~~functional~~, which corresponds to the talik total energy flux in relation to permafrost thaw. This study directly and analytically solves the Euler-Lagrange equation based on the stationary action principle rather than the entropy functional used in the phase field method.

I do not understand this description, and therefore cannot evaluate its validity – additional references would be helpful, but overall I think an expert in Lagrangian mechnanics is needed to understand if this method is appropriate in this situation.

Thank you for identifying this limitation. This study introduces an uncommon mathematical technique in this field, which is a key novel aspect of the work. We have an applied mathematician, Professor Yamatani, in our co-author list. He verified its mathematical consistency and scientific soundness.

“ the talik total energy flux in relation to permafrost thaw” – *what does this mean? The total energy flux into the talik?*

Yes, “the total energy flux into the talik” is a reasonable way to think of it. We have revised this part accordingly (Line 145).

Also, Equation (15) uses the method of Lagrange multipliers which is a common approach in machine learning field, lately. We hope the name of the method helps for readers to understand the physical interpretation.

Definitely helpful, but some physical interpretation needed. The application of the model to the lake in question should include some of this translation of the theory to measurements.

The solution of the Euler-Lagrange Equation is already thermodynamically optimum. The function type was not adjusted for the observed talik (or lake).

- l 341 is the assumption that the radial thermal gradient is zero accurate? Other publications report much more rapid horizontal than vertical thaw (though my focus is discontinuous PF) see McClymont et al. Devoie et al. work at Scotty Creek. Please cite something or report thermal gradients to support this.

Thank you. We cited McClymont et al. (2013) and Devoie et al. (2021) to support the approximation that the inter-seasonal average of the horizontal thermal gradient is negligible (Line 376-377)

*This is not what was expected – these publications show that lateral thaw is *FASTER* not slower, and do not comment on the lateral gradient. This also belongs in the model assumptions list – it is critical to the representation of the system, and therefore should be clearly listed as a limitation of the model.*

The lower the thermal gradient, the faster thaw: this is because the outgoing heat conduction (∝ thermal gradient, see Equation 1) slows the thaw. As such, horizontal lateral thaw is faster. We added the clarification there: "As a result, lateral thaw is faster than vertical thaw due to less energy loss to horizontal heat conduction", on Line 379-380.

- l 371 the preceding discussion all hinges on the zero lateral gradient assumption - please highlight this otherwise it seems unlikely

Thank you. We revised the assumption statement as "... assuming all other properties and horizontal thermal gradient variation are equal..." on Line 400-403.

Are equal to what? The horizontal thermal gradient = 0, the other properties are fixed?

This should be "remain unchanged". This part was revised accordingly on Line 406-407.

- l 535 what about anisotropic thermal properties? Maybe also discuss these as well?

Sorry. We could not catch it. Thermal properties (e.g. thermal conductivity, latent heat, and thawing temperature) should be isotropic, constant, and uniform. However, the thermal field is anisotropic (e.g. vertical temperature gradient).

Yes, this is your assumption, but is it reflected in reality? How may this be a source of error in your model's prediction?

Anisotropic thermal properties may be caused by consolidation of thawed porous media and the layered structure of the permafrost. Although the isotropic thermal properties are well-accepted and common assumptions for many existing thermal models as well as the Stefan Equation, we added a short paragraph on Line 577-581:

> Lastly, the 3D Stefan equation assumes fully saturated talik and isotropic thermal properties. However, uneven consolidation of thawed porous media and layered structure of the permafrost may cause anisotropic thermal conductivities. When the effect of direction dependency in the thermal conductivity is found to be significant, it can be incorporated into the formulation of the anisotropic heat conduction into the permafrost.