

# Non-normal flow rules affect fracture angles in sea ice viscous-plastic rheologies

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**Abstract.** The standard viscous-plastic (VP) sea ice model with an elliptical yield curve and a normal flow rule has at least two issues. First, it does not simulate fracture angles below  $30^\circ$  in uni-axial compression, in stark-contrast with observations of Linear Kinematic Features (LKF) in the Arctic Ocean. Second, there is a tight, but unphysical coupling between the fracture angle, post-fracture deformation, and the shape of the yield curve. This tight coupling was identified as the reason for the overestimation of fracture angles. In this paper, ~~we remove these issues are addressed by removing~~ the normality constraint on the flow rule in the standard VP model ~~and study its impact on the fracture angle in a simple~~. The new rheology is tested in numerical uni-axial compressive loading test loading tests. To this end, ~~we introduce a plastic potential independent of the yield curve that an elliptical plastic potential — which~~ defines the post-fracture deformations, or flow rule ~~— is introduced independently of the elliptical yield curve. As a consequence, the post-fracture deformation is decoupled from the mechanical strength properties of the ice.~~ ~~The numerical experiments show that the fracture angle strongly depends on the flow rule details.~~ We adapt the Roscoe's angle theory, which is based on observations of granular materials, to the context of sea ice modeling. In this framework, the fracture angles depend on both yield curve and plastic potential parameters. This new formulation predicts accurately the results of the numerical experiments with a root-mean-square error below  $1.3^\circ$ . The new rheology allows for angles of fracture smaller than  $30^\circ$  in uni-axial compression. For instance, a plastic potential with an ellipse aspect ratio smaller than ~~that of the standard ellipse gives fracture angles that are two~~ (i.e., the default value in the standard viscous-plastic model) can lead to fracture angles as low as  $22^\circ$ . ~~A newly adapted theory — based on one developed from observations of granular material — predicts numerical simulations of the fracture angles for plastic materials with a normal or non-normal flow rule with a root-mean-square error below 1.3.~~ Implementing an elliptical plastic potential in the standard VP sea ice model requires only ~~minor modifications~~. The small modifications to the standard VP rheology. The momentum equations with the modified rheology, however, takes longer are more difficult to solve numerically ~~for a fixed level of numerical convergence.~~ ~~In conclusion, the use of a plastic potential addresses several issues with the standard VP rheology : the fracture angle can be reduced to values~~ The independent plastic potential solves the two issues with VP rheology addressed in this paper: in uni-axial loading experiments, it allows for smaller fracture angles, which fall within the range of satellite observations and it can be decoupled from the exact, and it decouples the angle of fracture and the post-fracture deformation from the shape of the yield curve. ~~Furthermore, a different plastic potential function will be required to change~~ The orientation of the post-fracture deformation along the fracture lines (convergence ~~or divergence~~) ~~and to make the fracture angle independent on the~~

~~confining pressure (as in observations), and divergence~~, however, is still controlled by the shape of the plastic potential and the location of the stress state on the yield curve. A non-elliptical plastic potential would be required to change the orientation of deformation and to match deformation statistics derived from satellite measurements.

## 30 1 Introduction

Sea ice plays a significant role in the energy budget of the climate system and therefore has a strong influence on future climate projections. Sea ice dynamics are dominated by Linear Kinematic Features (LKFs), narrow lines of deformation that are observed in the Arctic sea ice cover, and also emerge in high-resolution simulations (~~Kwok, 2001; Hutchings et al., 2005~~). ~~LKFs in~~ (Kwok, 2001; Hutchings et al., 2005; Hutter et al., 2018; Koldunov et al., 2019; Heorton et al., 2018). LKFs can form in divergence, creating stretches of open water or leads, or in convergence, creating piles of ice or ridges (Stern et al., 1995). LKFs in the Arctic sea ice cover influence the Earth system in many ways: heat and matter exchange take place primarily over open water (Badgley, 1965). ~~Salt, salt~~ rejection during ice formation in leads creates dense water and influences the thermohaline circulation (~~Nguyen et al., 2011, 2012; Itkin et al., 2015~~). ~~The ice strength locally~~ (Nguyen et al., 2011, 2012; Itkin et al., 2015). Locally, the ice strength depends on the ~~ice thickness, sea ice state (e.g., thickness, concentration)~~, which in turn is affected by sea ice fracture with thermodynamical growth in opening-opened leads and with local dynamical growth during ridge formation.

In granular media like sea ice (Overland et al., 1998), deformation is localized along ~~pairs of~~ LKFs (Anderson, 1942; Erlingsson, 1988). Note, that in this study, we consider sea ice to be ~~granular of granular nature~~ not only in the marginal ice zone, but also in pack ice, where ice floes are densely packed. For this reason, we can consider the creation of an LKF as a process that involves both fracture and friction (Wilchinsky and Feltham, 2011). The intersection angles between the LKFs ~~influence have an influence on~~ the deformation field ~~; thus, and, hence, on~~ the local sea ice strength and the emergent sea ice anisotropy (Aksenov and Hibler, 2001). This anisotropy, which emerges as sea ice develops weak and strong areas along LKFs as leads or ridges form locally, then influences future deformation and fracture line orientation ~~, which will have impacts with effects~~ on the local sea ice mass balance. Therefore, reproducing the LKFs patterns, density, and orientation is important for accurate sea ice and climate projections at high-resolution.

This paper focuses on VP rheologies. Different models represent sea ice dynamics with different relations between stress and strains, for example, the Viscous-Plastic (VP, Hibler, 1977), Elastic-Plastic (EP, Coon et al., 1974), Elastic-Anisotropic-Plastic (EAP, Tsamados et al., 2013), or Maxwell-Elasto-Brittle (MEB, Dansereau et al., 2016) rheologies. In these different classes of models, various rheologies can be specified. In a VP rheology, a yield curve and plastic potential (flow rule) must be defined. The yield curve defines the stress criteria for the transition from small viscous deformations to large plastic deformations. The plastic potential determines the ensuing post-fracture deformation, called the flow rule. The flow rule is normal to the plastic potential (Drucker and Prager, 1952). The plastic potential can be set independently, or be equal to the yield curve. In the latter case, the flow rule is also perpendicular to the yield curve and is called a normal-flow rule or associated flow rule. Several yield curves have been used in sea ice models, some with a normal flow rule (Hibler, 1979; Zhang and Rothrock, 2005) and

60 some with a non-normal flow rule (Ip et al., 1991; Tremblay and Mysak, 1997; Hibler and Schulson, 2000; Wang, 2007). It is important to note that two models with the same material properties sharing the same yield curve but with different flow rules are two different rheologies, and may behave differently during the creation of LKFs.

LKFs have been studied ~~for several decades using~~ with observations (Stern et al., 1995; Kwok, 2001; Schulson and Hibler, 2004; Weiss et al., 2007) and numerical models (Spren et al., 2017; Hutter et al., 2018). In viscous-plastic (VP) sea ice  
65 models, LKFs are created because the ice is modeled as ~~a highly an extremely~~ viscous material between narrow zones of plastic deformation (Hutchings et al., 2005). This ~~behaviour~~ behavior has been argued to be the reason ~~why for low~~ temporal intermittency and spatial localization ~~are low~~ in VP models, leading to a spatial and temporal scaling of LKFs that is different from observations (Rampal et al., 2016). New models have been designed to represent sea ice fracture, for example, brittle models with a damage parameter that keeps the memory of previous fracture (~~Rampal et al., 2016; Dansereau et al., 2016~~)  
70 (~~Dansereau et al., 2016; Rampal et al., 2016; Girard et al., 2011~~), or anisotropic viscous-plastic rheologies models (Wilchinsky and Feltham, 2006; Heorton et al., 2018). Still, as of today, the viscous-plastic rheology with elliptical yield curve and normal flow rule (Hibler, 1979) is the *de facto* standard rheology. For example, of the 33 ~~Global Climate Models from~~ global climate models of the Climate Model Inter-comparison Project 5 (CMIP5), 30 use the standard VP rheology with an elliptical yield curve and normal flow rule (Stroeve et al., 2014).

75 Some of the ~~criticised issues of above issues related to~~ the standard VP rheology have already been addressed. For ~~example instance~~, high-resolution VP models also reproduce observed intermittency and spatial localization, even without brittle fracture dynamics; and the origin of intermittency remains an open question (~~Bouchat and Tremblay, 2017; Hutter et al., 2018; Bouchat et al., 2020, submit~~)  
(~~Bouchat and Tremblay, 2017; Hutter et al., 2018~~). However, the distribution of ~~intersection angles between LKFs in the Radarsat Geophysical Processor System (RGPS) dataset and the angles of fracture — defined as half of the intersection angle — in the~~  
80 ~~standard sea ice VP model with an elliptical yield curve with ellipse ratio  $e = 2$  (Hibler, 1979) dot not agree (Hutter et al., 2019) and the Radarsat Geophysical Processor System (RGPS) dataset still do not agree (Hutter et al., 2019; Hutter and Losch, 2020)~~.

The orientation of LKFs ~~has been the subject of many studies is a well studied subject~~ in the field of engineering and granular materials (LKFs are called shear bands in this field). Two classical solutions coexist and set two limit angles for the orientation  
85 of fractures: the Coulomb angle (static ~~behaviour~~ behavior) and the Roscoe angle (dynamic ~~behaviour~~ behavior). The Coulomb angle of fracture  $\theta_C$  between the fracture line and the first principal stress is determined ~~from the Mohr-Coulomb by the Mohr-Coulomb~~ criterion. It is a function only of the internal angle of friction  $\phi$  (Coulomb, 1773; Mohr, 1900):

$$\theta_C = \frac{\pi}{4} - \frac{\phi}{2}. \quad (1)$$

Roscoe (1970) challenged ~~the coulombic theory this concept~~ by considering the case of dilatant material and found from  
90 experiments with sand that the dilatancy angle  $\delta$  is the main parameter determining the orientation of shear bands (see Fig. 6 in Tremblay and Mysak, 1997, for a definition of the dilatancy angle  $\delta$  in the context of sea ice modeling.). Dilatancy refers to divergence along shear bands or LKFs. This divergence is a function is a function of the distribution of contact points between individual floes at the sub-grid scales. A positive angle of dilatancy is associated with contact points that (on average) oppose

95 the macroscopic shear motion and create divergence along the shear band; while negative dilatancy is associated with a closing of the shear line (ridging in the case of sea ice). The Roscoe angle of fracture is defined as:

$$\theta_R = \frac{\pi}{4} - \frac{\delta}{2}. \quad (2)$$

A general theory derived from experiments with sand that takes into account both the angle of friction and the angle of dilation combines the Coulomb and Roscoe angles as (Arthur et al., 1977; Vardoulakis, 1980):

$$\theta_A = \frac{\pi}{4} - \frac{1}{4}(\phi + \delta). \quad (3)$$

100 where  $\theta_A$  is called the Arthur angle. Tremblay and Mysak (1997) used this general theory to design their sea ice rheology. Vermeer (1990) proposed a theoretical framework based on the grain size and showed that the angle of fracture in most experiments lie falls between the two extremes:  $\theta_C \leq \theta \leq \theta_R$ , with  $\delta < \phi$  in granular materials like sandsands. If  $\phi = \delta$  then  $\theta_R = \theta_C = \theta_A$ , and the flow rule is normal to the yield curve. In other words, for  $\phi = \delta$ , the principal axes of stress and the principal axes of strain are coaxial. This condition, however, is not generally satisfied for granular materials (Balendran and Nemat-Nasser, 1993). Experiments with sand have shown differences between  $\phi$  and  $\delta$  of the order of  $30^\circ$  (Vardoulakis and Graf, 1985; Bolton, 1986). Note that both mechanisms, friction and dilatancy, are not radically different: a larger dilatancy angle implies a larger grain size, more contact normals opposing the flow, hence more friction ~~-(Ringeisen et al. (2019) used the (Vermeer, 1990). The~~ theory of the internal angle of friction can be used to link fracture angles to rheologies with a normal flow rule ~~in their appendix B to link fracture angles to rheology. (Ringeisen et al., 2019, Appendix B). The effects of a non-normal~~ 110 flow rule for sea-ice rheologies (as in e.g., Hibler and Schulson, 2000; Hutchings et al., 2005) on the fracture angles have not been explored. Therefore, it is unknown which of the three theories (Coulomb, Roscoe, Arthur) provide the most accurate prediction for this case.

The fracture angles with the standard sea-ice-VP rheology cannot be smaller than  $30^\circ$  in uni-axial compression, even by changing the ellipse aspect ratio  $e$ , as shown by idealized experiments and theory (Ringeisen et al., 2019). This minimum 115 angle is in conflict with observations that report fracture angles (~~half of the intersection angles~~) generally below  $30^\circ$ . Observations report fracture angles of  $14^\circ$  (Marko and Thomson, 1977),  $15 \pm 1.5^\circ$  (Erlingsson, 1988),  $17^\circ$  to  $18^\circ$  (Cunningham et al., 1994),  ~~$24^\circ$  ( $\mu = 0.9$ , Weiss and Schulson, 2009), and~~ and a clear peak in the distribution of angles between  $20^\circ$  to  $25^\circ$  (Hutter and Losch, 2020). Further, uni-axial compression experiments (Ringeisen et al., 2019) showed that: (1) the angle of fracture is a function of the gradient of shear strength with respect to compressive strength ~~set by the ellipse aspect~~ 120 ratio (i.e., the slope of the yield curve), (2) the ellipse aspect ratio determines the divergence along the LKFs, and (3) the fracture angle is a function of the confining pressure. In Ringeisen et al. (2019), the confinement was achieved by adding thinner ice on either side of an ice slab subjected to uni-axial loading. These three properties of the standard VP rheology do not comply with the theory and observations of granular media behaviourbehavior, namely that shear band orientations and divergent ~~for~~ convergent motion at the slip lines are a function solely-mainly of the shear strength of the material and 125 orientation of the contact normals (or ~~dilatation angle~~dilatancy angle), and that the confining pressure has only a limited effect (Alshibli and Sture, 2000; Han and Drescher, 1993; Desrues and Hammad, 1989). This unphysical behaviourbehavior of the standard VP rheology is connected to the shape of the yield curve in conjunction with a normal flow rule.

The flow rule has the advantage that it can be observed with remote sensing methods, ~~contrary to stress which need in~~ contrast to observing stress which requires in-situ measurements. The ratio of shear ~~and to~~ divergence along the shear bands or LKFs allows to infer the dilatancy angle. ~~Observations of granular material. Observations of sea ice drift in the Arctic~~ show that most of the deformation takes place in shear, ~~with that is,~~ 98% of ~~deformations showing deformation show~~ more shear than divergence or convergence (Stern et al., 1995). The ellipse ratio of the standard model can be modified to fit this distribution (Bouchat and Tremblay, 2017). ~~Also, laboratory experiments with first-year ice showed that flow rules are non-normal during brittle failures (Weiss et al., 2007).~~ Separating the link between the fracture angle and the flow rule from the yield curve is necessary to design VP rheologies that are consistent with observed sea ice deformation.

~~This paper focuses on VP rheologies. Different models represent sea ice dynamics with different material properties, for example, Viscous-Plastic (VP, Hibler, 1977), Elastic-Plastic (EP, Coon et al., 1974), or Maxwell-Elasto-Brittle (MEB, Dansereau et al., 2017). In these different classes of models, various rheologies can be defined. In a VP rheology, a yield curve and plastic potential (flow rule) must be defined. The yield curve defines the stress criteria for the transition from small viscous deformations to large plastic deformations. The plastic potential determines the ensuing post-fracture deformation, called the flow rule. The flow rule is normal to the plastic potential (Drucker and Prager, 1952). The plastic potential can be set independently, or be equal to the yield curve. In the latter case, the flow rule is also perpendicular to the yield curve and is called a normal flow rule or associated flow rule. Several yield curves have been used in sea ice models, some with a normal flow rule (Hibler, 1979; Zhang and Rothrock, 2005) and some with a non-normal flow rule (Ip et al., 1991; Tremblay and Mysak, 1997; Hibler and Schulson, 2000; Wang, 2007). It is important to note that two models with the same material properties sharing the same yield curve but with different flow rules are two different rheologies.~~

In this paper, ~~we investigate the effects of a~~ we investigate the effects of a non-normal flow rule on fracture angles. We use the non-normal flow rule on fracture angles and present a generalized framework to use with flow rule as a means of separating the state of stress (at failure) and the post-fracture deformation. To this end, we study the non-normal flow rule in the context of the standard VP rheological model using a similar shape for the plastic potential (i.e., an ellipse) because (1) the ellipse is widely used in the community, and (2) its behavior is well documented (compared to other models), providing a solid basis for comparison. For these two reasons, we use the elliptical yield curve despite the fact that it is not the most appropriate yield curve to model sea ice as a granular material like sea ice. This paper provides a new generalized theoretical framework for any viscous-plastic material with any flow rules (normal or non-normal). ~~To this end, we introduce a plastic potential independent from yield curve. The new model is tested flow rules. Following Ringeisen et al. (2019), we test the new model~~ in simple uni-axial loading experiments where the relationship between fracture angle and flow-rule can be easily identified.

The paper is structured as follows. Section 2 describes the model (2.1), the new rheology (2.2), and a general theory linking the fracture angles and a general flow rule (2.3). The sections 3 and 4 describe the idealized experimental setup and the results. Section 5 discusses these results and their implication on current and future rheologies. Conclusions follow in section 6.

### 2.1 Building the sea ice VP constitutive equations

We consider sea ice as a 2D viscous-plastic material. The ice velocities are calculated from the sea ice momentum equations:

$$\rho h \frac{\partial \mathbf{u}}{\partial t} = -\rho h f \mathbf{k} \times \mathbf{u} + \boldsymbol{\tau}_a + \boldsymbol{\tau}_o - \rho h \nabla \phi(0) \phi_s + \nabla \cdot \boldsymbol{\sigma}, \quad (4)$$

where  $\rho$  is the ice density,  $h$  is the grid cell averaged sea ice thickness,  $\mathbf{u}$  is the ice drift velocity field,  $f$  is the Coriolis parameter,  $\mathbf{k}$  is the vertical unit vector,  $\boldsymbol{\tau}_a$  is the surface air stress,  $\boldsymbol{\tau}_o$  is the ocean drag,  $\nabla \phi(0) \phi_s$  is acceleration from the gradient of sea surface height, and  $\boldsymbol{\sigma}$  is the vertically integrated internal ice stress tensor defined by the sea ice VP constitutive equations. The constitutive equations ~~link the stress tensor  $\boldsymbol{\sigma}$  to the deformation tensor  $\dot{\boldsymbol{\epsilon}}$~~  define the vertically integrated stress tensor  $\boldsymbol{\sigma}$  as function of the strain rate tensor  $\dot{\boldsymbol{\epsilon}}$  and the state variables  $\chi$  (e.g., ice thickness, ice strength, ice concentration, etc.). The components of the strain rate tensor are computed from the velocities as  $\dot{\epsilon}_{ij} = \frac{\partial u_i}{\partial x_j}$ . The constitutive equations then have the form:

$$\boldsymbol{\sigma} = f(\dot{\boldsymbol{\epsilon}}, \chi). \quad (5)$$

In an ideal plastic model, the stresses are independent of the strain rates; in the VP model the stresses are independent of the strain rates for large deformation events (the plastic states with stresses on the yield curve) and they depend on the strain rates for small deformations (the viscous states with stresses inside the yield curve). It is this set of equations that defines the rheology of sea ice and determines the fracture pattern and the opening or closing along the fractures.

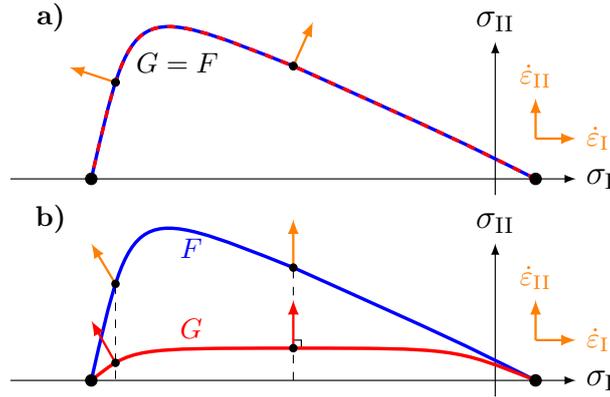
One of the state variables in the model is the ~~maximal~~ maximum compressive strength  $P$ . This variable represents the ~~maximal~~ maximum compressive stress that sea ice can bear in uniform compression before ridging. We use the simple standard relationship (Hibler, 1979):

$$P = P^* h e^{-C^*(1-A)}, \quad (6)$$

where  $C^*$  is a free parameter (typically  $C^* = 20$ ),  $h$  is the mean ice thickness,  $A$  is the fractional sea ice area cover in a grid cell, and  $P^*$  is the ice strength of 1 m ice at 100% concentration ( $A = 1$ ). Some other state variables are a function of  $P$ ; for instance, the tensile strength  $T$  is usually defined as  $T = k_t \cdot P$ , where the tensile factor  $k_t > 0$  (König Beatty and Holland, 2010). Others are not, such as the ellipse aspect ratio (Hibler, 1979) or the internal angle of friction (Ip et al., 1991). The equation for the yield curve in a VP model is written in terms of the state variables.

For two-dimensional sea ice, stress is a rank two tensor; thus, it has four components. The yield curve represents the stress states ~~that are deforming for which sea ice deforms~~ plastically while enclosing the states of stress for slow viscous deformations ~~stress states for which sea ice slowly deforms viscously~~. We express the yield curve as a function of the stresses  $\sigma_{ij}$  and the state variables  $\chi$ :

$$F(\sigma_{ij}, \chi) = 0. \quad (7)$$



**Figure 1.** Schematic yield curve  $F$  (blue) and plastic potential  $G$  (red) for a normal (a) and non-normal (b) flow rule. The flow rule (orange) for a given stress on the yield curve is normal to the plastic potential (red) for the same  $\sigma_I$ . Note that the stress and strain invariant axes are assumed to coincide.

190 The yield curve can be represented in principal stress ( $\sigma_1$  and  $\sigma_2$ ) or stress invariants space ( $\sigma_I$  and  $\sigma_{II}$ ). Figure 1 shows an arbitrary yield curve in stress invariants space. Although equation (7) determines if the deformation is plastic or viscous, it does not determine how the ice will deform after fracture. In order to obtain a closed system of equations, we define a plastic potential that defines the flow rule.

The plastic potential determines the direction of deformation for stress states on the yield curve. The flow rule represents the direction of deformation in the grid cell. The orientation of the flow rule in the coordinate system ( $\dot{\epsilon}_I, \dot{\epsilon}_{II}$ ), as shown in orange in Fig. 1, indicates if the grid cell deforms in convergence ( $\dot{\epsilon}_I < 0$ ) or divergence ( $\dot{\epsilon}_I > 0$ ), and shear ( $\dot{\epsilon}_{II}$ ). Just as the yield curve, the plastic potential can be written as:

$$G(\sigma_{ij}, \chi) = 0. \quad (8)$$

200 The direction of the deformation, called the flow rule, is perpendicular to the plastic potential. This is shown in red on Fig. 1b and mathematically expressed by

$$\frac{\partial G}{\partial \sigma_{ij}}(\sigma_{ij}, \chi) = \lambda \dot{\epsilon}_{ij}, \quad (9)$$

where  $\lambda > 0$  is the unknown flow rate. The flow rule is applied for stress states on the yield curve at the same compressive stresses (orange arrows in Fig. 1b). If the plastic potential and the yield curve are the same ( $G = F$ ), the flow rule is called an associative or normal flow rule, as the flow rule is also perpendicular to the yield curve (see Fig. 1a).

205 Using Eq. (7) and Eq. (9), we can write a system of 5 equations (four from Eq. 9 and one from Eq. 7) for 5 unknowns ( $\sigma_{11}, \sigma_{22}, \sigma_{12}, \sigma_{21}, \lambda$ ). Solving this system of equations allows us to write the constitutive equations for the sea ice model as function of the components of the strain-rate ( $\dot{\epsilon}_{11}, \dot{\epsilon}_{22}, \dot{\epsilon}_{12}, \dot{\epsilon}_{21}$ ) and the state variables  $\chi$ .

After deriving these constitutive equations, we assume that the stress and strain rate tensors are symmetric, that is,  $\sigma_{12} = \sigma_{21}$  and  $\dot{\epsilon}_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$ . The symmetry follows from ignoring the rotation in an isotropic medium. Note that we first  
 210 need to solve this system of equations without using the symmetry condition: the symmetry condition is only invoked at the end. Applying the symmetry before solving the system of equation changes the nature of the initial tensor, and the resulting constitutive equations would be different.

An ideal plastic model, with the stresses independent of the strain rates, has a singularity because the non-linear viscosities tend to infinity as the strain rates tend to zero. Hibler (1977) solved this issue with a regularization that limits the value of  
 215 the bulk and shear viscosities  $\zeta$  and  $\eta$  to a maximum value. When the viscosities are capped to their maximum values, the stresses are linearly related to the strain rates and the material behaves as a viscous material. [VP sea-ice models typically cap the viscosity at](#)

$$\zeta_{\max} = \frac{1}{2\Delta_{\min}} \cdot P = (2.5 \times 10^8 \text{ s}) \cdot P \quad (10)$$

and  $\eta_{\max} = \frac{\zeta_{\max}}{e_C}$  to regularize the momentum equations. When this regularization is in effect,  $\zeta$  and  $\eta$  are independent of the  
 220 deformation field ( $\Delta$ ) and the stress divergence reduces to harmonic viscosity with constant coefficients.  $\Delta_{\min} = 2 \times 10^{-9} \text{ s}^{-1}$  (Hibler, 1979, 1977) translates to a deformation time scale of almost 16 years. Therefore, viscous deformations are slow and negligible with respect to the plastic deformations that operate on synoptic time scale, and VP rheologies can be considered as ideal plastic. The viscous behavior can be seen as a consequence of regularizing the viscosities rather than an implementation of a physical behavior.

## 225 2.2 Elliptical yield curve with non-normal flow rule

We now build a rheology with an elliptical yield curve and a non-normal flow rule, that is, we use a plastic potential  $G$  that is different from the yield curve  $F$ . [By doing this, we will change the orientation of the flow rule, without changing the yield stress state \(see Fig. 2 and Fig. 4 for some examples\).](#) We use a different, but still elliptical plastic potential for simplicity: this choice **only** requires only minor modifications to a typical VP sea ice model. We define the yield condition  $F$  and the  
 230 plastic potential  $G$  as a function of the state variables  $\chi$ : the ice compression strength  $P$ , the ice tensile strength  $T = k_t P$  (König Beatty and Holland, 2010), the yield curve's ellipse ratio  $e_F$ , and the plastic potential's ellipse ratio  $e_G$  by

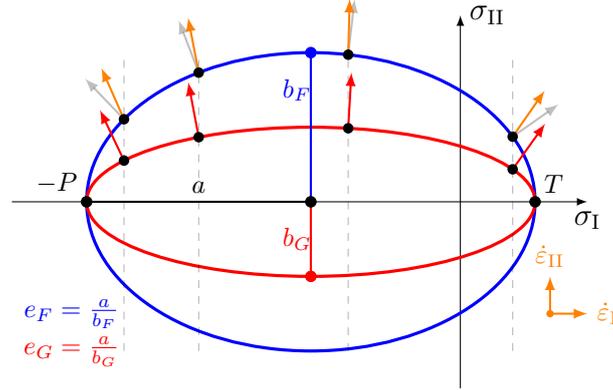
$$X(\sigma_I, P, e_X, k_t) = \left( \frac{\sigma_I + \frac{P(1-k_t)}{2}}{\frac{P(1+k_t)}{2}} \right)^2 + \left( \frac{\sigma_{II}}{\frac{P(1+k_t)}{2e_X}} \right)^2 - 1 = 0, \quad (11)$$

for  $X = F, G$  for the yield curve or the plastic potential. Using Eq. (11), we write  $\sigma_{II}$  as a function of  $\sigma_I$  as:

$$\sigma_{II, X} = \frac{1}{e_X} \sqrt{P^2 k_t - \sigma_I^2 - \sigma_I P (1 - k_t)}. \quad (12)$$

235 Following Hibler (1977, 1979), we derive the constitutive equations  $\sigma_{ij}$ :

$$\sigma_{ij} = 2\eta \dot{\epsilon}_{ij} + (\zeta - \eta) \dot{\epsilon}_{kk} \delta_{ij} - \frac{P(1-k_t)}{2} \delta_{ij}, \quad (13)$$



**Figure 2.** Elliptical yield curve with a non-normal flow rule, a yield curve ellipse aspect ratio  $e_F = 2$  (blue) and a plastic potential ellipse aspect ratio  $e_G = 4$  (red). The gray and orange arrows show the normal and non-normal flow rules, respectively.

where the shear and bulk viscosities  $\eta$  and  $\zeta$  are defined by:

$$\zeta = \frac{P(1+k_t)}{2\Delta} \text{ and } \eta = \frac{\zeta}{e_G^2} = \frac{P(1+k_t)}{2e_G^2\Delta} \quad (14)$$

with

$$\Delta = \sqrt{(\dot{\epsilon}_{11} - \dot{\epsilon}_{22})^2 + \frac{e_F^2}{e_G^4}((\dot{\epsilon}_{11} - \dot{\epsilon}_{22})^2 + 4\dot{\epsilon}_{12}^2)}. \quad (15)$$

Figure 2 shows an example of yield curve and plastic potential, with the resulting flow rule. For  $e_G > e_F$ , the absolute value of the divergence is smaller and the shear strain rate is larger compared to a normal flow rule ( $e_G = e_F$ ) and vice versa for  $e_G < e_F$ .

### 2.3 Linking fracture and flow rule

In this section, we generalize the theory linking the rheological model and the fracture angles in simple uni-axial compressive test (Ringeisen et al., 2019) to materials with a non-associated flow rule. To this end, we follow the theory of Roscoe (1970) where the angle of fracture depends uniquely on the angle of dilatancy of a granular material. Based on laboratory experiments, Roscoe (1970) states that the velocity characteristics (the post-failure deformation) seem to be a better predictor than the stress characteristics (the stress at failure) for the orientation of shear bands in granular materials.

Figure 3 illustrates the case of an arbitrary yield curve with an arbitrary plastic potential. The figure To adapt the Roscoe angles to sea ice modeling, we proceed as follows: (1) the stress state on the yield curve (point  $\mathbf{p}$  on Fig. 3a) defines the position and size the Mohr's circle at fracture (blue circle on Fig. 3b), (2) the slope of the plastic potential determines the point on the Mohr's circle where deformation takes place, that is, the slope directly predicts the fracture angle  $\theta$  as a function of the dilatancy angle  $\delta$  (per Roscoe theory, Fig. 3b). For the special case of uni-axial compression, we (A) determine the stress

255 state on the yield curve for uni-axial compression as a function of the yield curve ellipse ratio  $e_F$ , and (B) compute the slope of the plastic potential at that stress state as a function of the plastic potential ellipse ratio  $e_G$ . Finally, we combine (2) and (B) to compute the theoretical prediction for the fracture angle as a function of ellipse ratios  $e_G$  and  $e_F$ . Figure 3 shows the geometrical construction that links the angle of dilatancy  $\delta$  to the slope of the plastic potential  $\tan(\gamma_G)$ :

$$\sin(\delta) = \tan(\gamma_G) = -\frac{\partial\sigma_{II,G}}{\partial\sigma_I}. \quad (16)$$

260 Note that the minus sign above was included in the derivative of the yield curve function in Eq. (B1) and (B2) of Ringeisen et al. (2019). This equation agrees with the definition of Roscoe (1970)  $\sin(\delta) = \frac{\dot{\epsilon}_I}{\dot{\epsilon}_{II}}$ , because the ratio of  $\dot{\epsilon}_I$  to  $\dot{\epsilon}_{II}$  is equal to the slope of the plastic potential  $-\frac{\partial\sigma_{II,G}}{\partial\sigma_I}$ , as the flow rule is perpendicular to the plastic potential. Figure 3 also shows the normal flow rule, which, in agreement with the coulombic theory, would lead to different fracture angles (light blue lines). From Fig. 3, the fracture angle can be written as:

$$265 \theta_R = \frac{\pi}{4} - \frac{\delta}{2}. \quad (17)$$

Substituting Eq. (16) in the equation above, the relationship between the fracture angle and the plastic potential becomes

$$\theta(\sigma_I) = \frac{1}{2} \left[ \frac{\pi}{2} - \arcsin \left( -\frac{\partial\sigma_{II,G}}{\partial\sigma_I}(\sigma_I) \right) \right] = \frac{1}{2} \arccos \left( -\frac{\partial\sigma_{II,G}}{\partial\sigma_I}(\sigma_I) \right).$$

$$\theta_R(\sigma_I) = \frac{1}{2} \left[ \frac{\pi}{2} - \arcsin \left( -\frac{\partial\sigma_{II,G}}{\partial\sigma_I}(\sigma_I) \right) \right] \quad (18)$$

$$270 = \frac{1}{2} \arccos \left( -\frac{\partial\sigma_{II,G}}{\partial\sigma_I}(\sigma_I) \right). \quad (19)$$

We calculate the fracture angles for the elliptical yield curve with non-normal flow rule in uni-axial compression along the  $y$  axis. In this case,  $\sigma_{11} = \sigma_{12} = 0$ ,  $\sigma_{22} < 0$ , and the principal stresses and stress invariants can be written as:

$$\sigma_1 = \frac{1}{2} \left( \sigma_{11} + \sigma_{22} + \sqrt{(\sigma_{11} - \sigma_{22})^2 + 4\sigma_{12}^2} \right) = 0, \quad (20)$$

$$275 \sigma_2 = \frac{1}{2} \left( \sigma_{11} + \sigma_{22} - \sqrt{(\sigma_{11} - \sigma_{22})^2 + 4\sigma_{12}^2} \right) = \sigma_{22}. \quad (21)$$

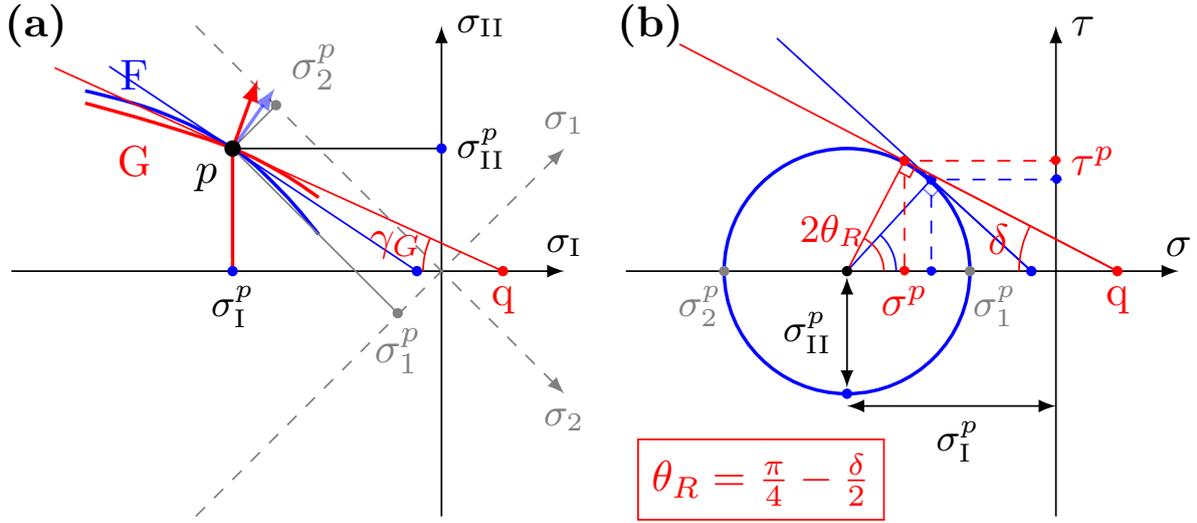
$$\sigma_I = \frac{\sigma_1 + \sigma_2}{2} = \frac{\sigma_{22}}{2} \quad (22)$$

$$\sigma_{II} = \frac{\sigma_1 - \sigma_2}{2} = -\frac{\sigma_{22}}{2} = -\sigma_I. \quad (23)$$

From Eq. 23, the maximum shear stress  $\sigma_{II,F}^p$  in the fracture plane in uni-axial compression can be expressed as

$$\sigma_{II,F}^p(\sigma_I^p) = -\sigma_I^p, \quad (24)$$

280 where  $p$  indicates the stress state at the fracture. Figure 4 shows the stress trajectory in principal stress space for uni-axial compression. It also shows how the flow rule changes for the same stress state when using two different elliptical plastic potentials.



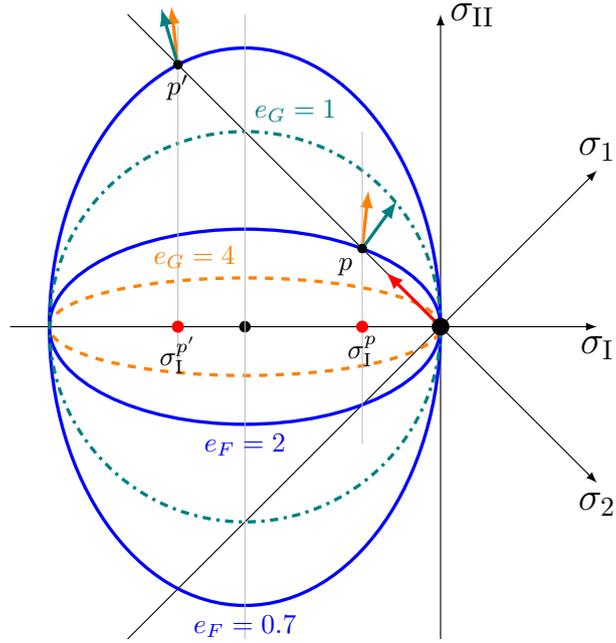
**Figure 3.** Link between fracture angle and yield curve: **a) (a)** Arbitrary yield curve  $F$  (blue) and plastic potential  $G$  (red) in stress invariants space. The plastic potential and yield curve intersect at a stress state  $p$  for illustration purposes only. The orange-red arrow is perpendicular to  $G$ , but non-normal to the yield curve  $F$ . The tangent to the plastic potential  $G$  at point  $p$  has a slope  $\mu_G = \tan(\gamma_G)$  and intersects the  $\sigma_I$ -axis at point  $q$  (thin red line). For reference, the normal and tangent to the yield curve  $F$  are shown as a thin blue arrow and line. **Mohr's circle of stress for this stress state (blue) has a radius of  $\sigma_{II}^p$  and centers on  $\sigma_I^p$  on the  $\sigma_I$ -axis.** Gray dashed lines show the principal stress axes. **b) (b)** Mohr's circle for the fracture state  $p$  in **a) (a)** (for normal in blue and for non-normal flow rule in red) in the fracture plane of reference  $(\sigma, \tau)$  of center  $\sigma_I^p$  and radius  $\sigma_{II}^p$ . The thin red line is the tangent to the Mohr's circle that passes passing through the point  $q$  on the  $\sigma$  axis. By this geometrical construction,  $\sin(\delta) = \tan(\gamma_G) = \mu_G$  (Only valid for  $|\mu_G| \leq 1$ ).  $\delta$  is called the dilatancy angle. Again for comparison with panel (a), the transparent blue lines show and dots depicts the corresponding construction for case of the normal flow rule (i.e.,  $G = F$ ). When considering the plastic potential, the angle of fracture is written as of the dilatancy angle  $\delta$  as  $\theta = \frac{\pi}{4} - \frac{\delta}{2}$ . By comparison again,  $\delta = \phi$  in the case of a normal flow rule from panel a) and the Roscoe theory ( $\theta = \frac{\pi}{4} - \frac{\phi}{2}$ ) reduces to the Mohr-Coulomb theory ( $\theta = \frac{\pi}{4} - \frac{\phi}{2}$ ).

In the following, we use the normalized stress invariants  $\sigma'_I = \frac{\sigma_I}{P}$  and  $\sigma'_{II} = \frac{\sigma_{II}}{P}$  to simplify the notation. The slope of the yield curve or the plastic potential depends only on  $e_F$  and  $e_G$ , but not on  $P$ . Substituting Eq. (12),  $\sigma'_I$ , and  $\sigma'_{II}$  in Eq. (24), we obtain,

$$\sigma'_{II}{}^p = -\sigma'_I{}^p = \frac{1}{e_F} \sqrt{k_t - \sigma'_I{}^p (\sigma'_I{}^p + 1 - k_t)}, \quad (25)$$

and solve the first stress invariant  $\sigma_I^p$  on the fracture plane in uni-axial compression

$$\sigma_I^p = \frac{(k_t - 1) - \sqrt{(1 - k_t)^2 + 4k_t(1 + e_F^2)}}{2(1 + e_F^2)}. \quad (26)$$



**Figure 4.** Trajectory of maximum normal stress (red arrow) in a uni-axial loading test experiment in a material with two different elliptical yield curves (blue) and plastic potentials (dashed orange and dash-dotted teal). The orange and teal arrows show the flow rule normal to the plastic potential of the same colour for the same stress state. For  $e_G < e_F$ , the ratio of divergence to shear increases. The opposite is true for  $e_G > e_F$ . A similar figure in principal stress space is presented in Ringenisen et al. (2019)

The slope of the tangent at  $\sigma_I^p$  to the plastic potential is given by the derivative of Eq. (12):

$$290 \quad \frac{\partial \sigma'_{II,G}}{\partial \sigma'_I}(\sigma_I^p) = \frac{1}{2e_G} \frac{-2\sigma_I^p - 1 + k_t}{\sqrt{k_t - \sigma_I^p(\sigma_I^p + 1 - k_t)}}. \quad (27)$$

Substituting Eq. (26) into Eq. (27), yields

$$\frac{\partial \sigma'_{II,G}}{\partial \sigma'_I} \Big|_{\sigma_I^p} = \frac{1}{e_G e_F} \left( 1 - \frac{(1 + e_F^2)}{1 + \sqrt{1 + 4 \frac{k_t}{(1 - k_t)^2} (1 + e_F^2)}} \right). \quad (28)$$

or for zero tensile strength ( $k_t = 0$ ),

$$\frac{\partial \sigma'_{II,G}}{\partial \sigma'_I} \Big|_{\sigma_I^p, k_t=0} = \frac{1}{2e_F e_G} (1 - e_F^2). \quad (29)$$

295 The fracture angle can finally be written as a function of  $e_G$  and  $e_F$  from Eq. (18):

$$\theta_{e,nn}(e_F, e_G) = \frac{1}{2} \arccos \left( \frac{1}{2e_F e_G} (e_F^2 - 1) \right). \quad (30)$$

As expected, for  $e_F = e_G = e$  we recover the fracture angle derived in Ringeisen et al. (2019):

$$\theta_{e,n}(e) = \frac{1}{2} \arccos \left[ \frac{1}{2} \left( 1 - \frac{1}{e^2} \right) \right]. \quad (31)$$

### 3 Experimental setup and numerical scheme

300 Following Ringeisen et al. (2019), we load a rectangular ice floe of 8 km by 25 km with a ~~thickness of  $h = 1$  m and a uniform~~ thickness of  $h = 1$  m and a uniform sea ice concentration of  ~~$A = 1.0$~~   $A = 100\%$  (see Fig. 5). The numerical domain has the dimensions  $L_x = 10$  km and  $L_y = 25$  km. At  $y = 0$ , we use a closed, solid boundary with a no slip condition (i.e.,  $u = v = 0$ ). At  $x = 0$  and  $L_x$ , we use Neumann boundary conditions:

$$\frac{\partial A}{\partial x} \Big|_{x=0, L_x} = \frac{\partial h}{\partial x} \Big|_{x=0, L_x} = \frac{\partial u}{\partial x} \Big|_{x=0, L_x} = \frac{\partial v}{\partial x} \Big|_{x=0, L_x} = 0. \quad (32)$$

305 On the left and right sides of the domain ( $x < 1$  km and  $x > 9$  km), we have open water between the ice floe and the boundary to ensure that the boundaries have no effect on the simulation. At ( $y = L_y$ ), we use a Dirichlet boundary condition for ice velocity ( $v$  the velocity in  $y$ -direction increasing linearly in time simulating an axial loading test) and a Neumann boundary condition for ice thickness and concentration :

$$\underline{v(t)|_{y=L_y} = a_v \cdot t, u(t)|_{y=L_y} = 0; \frac{\partial A}{\partial y} \Big|_{y=L_y} = \frac{\partial h}{\partial y} \Big|_{y=L_y} = 0}$$

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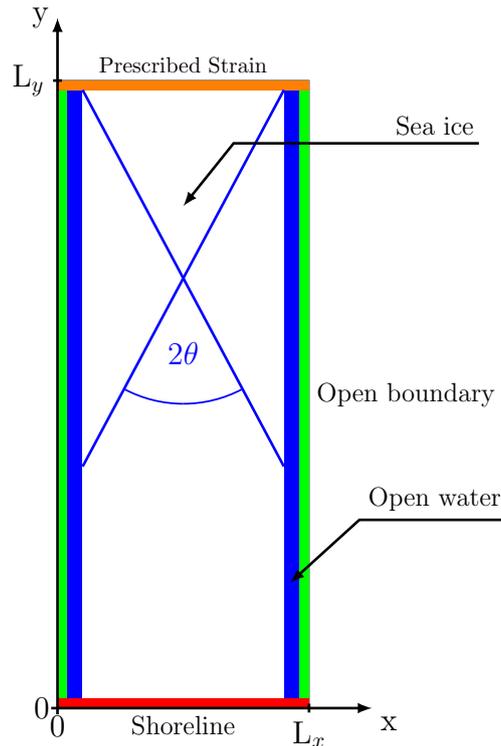
$$\underline{v(t)|_{y=L_y} = a_v \cdot t, u(t)|_{y=L_y} = 0} \quad (33)$$

$$\underline{\frac{\partial A}{\partial y} \Big|_{y=L_y} = \frac{\partial h}{\partial y} \Big|_{y=L_y} = 0} \quad (34)$$

with  $a_v = -5 \cdot 10^{-4} \text{ ms}^{-2}$ . The grid spacing of the domain is 25 m, and the timestep is 0.1 s. For simplicity, the Coriolis parameter  $f = 0$ .

315 The non-linear momentum equations (4) are integrated using a Picard solver with 15 000 non-linear (or outer-loop) iterations (Losch et al., 2010). For the linearized problem within each non-linear iteration, we use a line successive (over-)relaxation (LSR) method (Zhang and Hibler, 1997), with a tolerance criterion of  $|u_k - u_{k-1}|_{\max} < 10^{-11} \text{ ms}^{-1}$ , where  $k$  is the linear iteration index. We use an inexact approach with only a maximum of 200 linear iterations for the linearized equations; the linearized system does not reach the tolerance criterion for the first non-linear iterations, but does so as the non-linear system

320 approaches a converged solution. We chose a very small tolerance and residual norm for the solution of the linear and non-linear problem in order to simulate a clean fracture with a well defined fracture angle ~~—~~ — for comparison with theory and observations. These criteria are much stricter than common recommendations for Arctic sea ice simulations (e.g. Lemieux and Tremblay, 2009). We expect ~~modeling sea ice~~ that numerical sea ice models are computationally more challenging with a non-normal flow rule ~~to be more challenging~~ than with a normal flow rule. The ~~non-coaxiality of the deviatoric stress and~~



**Figure 5.** Model domain with a solid wall at  $y = 0$  (red), Dirichlet boundary conditions with  $u = 0$  at  $y = 0$  and prescribed velocities at  $y = L_y$ . Open boundaries at  $x = 0, L_x$  (green) with Neumann boundary conditions. For the conservation of mass, ice thickness and concentration equations ( $h, A$ ) Neumann boundary conditions are used on all boundaries.  $\theta$  is the measured fracture angle with respect to the vertical; the blue line represents an LKF.

325 ~~strain rate non-normality of the flow rule relative to the yield curve~~ introduces more complexity because Drucker's ~~postulate~~  
~~for stability is not respected~~ ~~stability postulate is not satisfied~~ (Vermeer and De Borst, 1984; Balendran and Nemat-Nasser,  
 1993). This particular uni-axial loading experiment is also complex to solve numerically because the forcing is localized on the  
 boundary, in contrast to real geophysical system integrations where wind and ocean currents are acting over the entire surface  
 of the ice.

330 The intersection angles between the LKFs are measured with the *Measure Tool* from the GNU Image Manipulation Program  
 (GIMP, version 2.8.16, gimp.org). The first 5 seconds of simulations are used to define the sea ice fracture and calculate the  
 fracture angle. The angle of each fracture lines is measured and used to compute the average fracture angle and the standard  
 deviation  $\sigma = \sqrt{\text{VAR}}$ . Note that the fracture angles do not depend on resolution, scale, geometry, or boundary conditions (see  
 Ringeisen et al., 2019, their Sec. 3.2 ). We do not use a replacement pressure scheme (Ip et al., 1991; Ip, 1993), because it has  
 335 no influence with the angle of fracture (not shown).

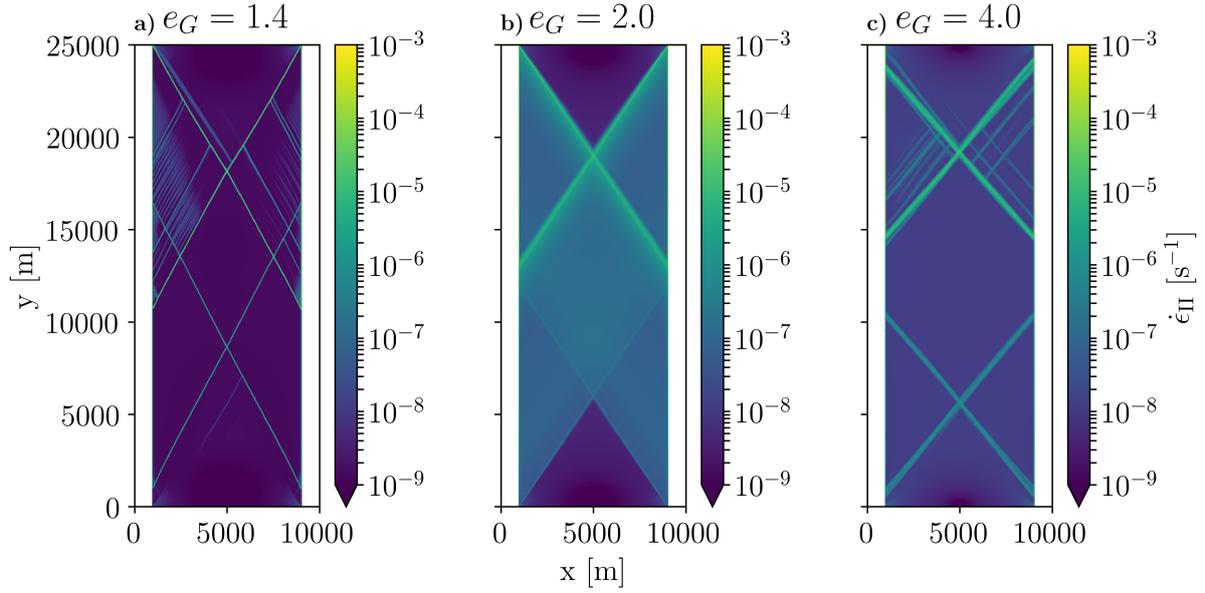
## 4 Results

We study the evolution of the fracture angle  $\theta$  when the plastic potential changes while the yield curve stays the same (see Fig. 4 for details). In this manner, the ice breaks for the exact same stress state but with a different flow rule. For simplicity, we test here the elliptical yield curve without tensile strength ( $k_t = 0$ ).

340 Figure 6 shows the fracture pattern for the standard yield curve ellipse ratio  $e_F = 2.0$  and three values of the plastic potential ellipse ratio  $e_G = 1.4, 2.0,$  and  $4.0$ . ~~Following observations and laboratory experiment, the ice fractures have~~ The fractures form a diamond shape ~~pattern (Erlingsson, 1988; Wilchinsky et al., 2010), similar to the shapes observed at large scales (Erlingsson, 1988), in laboratory experiments (Schulson, 2001), and modeled with DEM models (Wilchinsky et al., 2010) or other continuous sea ice models (Ringeisen et al., 2019; Heorton et al., 2018).~~ With a normal flow rule ( $e_G = 2.0$ ), single  
345 pairs of fracture lines with one unique fracture angle, large deformation along the LKFs, and smaller deformations (by several orders of magnitude) within diamond-shape floes are simulated. With the non-normal flow rule ( $e_G = 1.4$  and  $e_G = 4$ ) we make three observations:

1. Asymmetric secondary fracture lines appear, in contrast to the normal flow rule simulation. We attribute the asymmetry and presence of secondary fractures to the lack of full numerical convergence associated with the violation of Drucker's  
350 principle, or the non-normality of the flow rule (the ratio of divergence to shear strain rate differs from that of the shear to normal stress). For instance, the ~~grid-cell average residual norm~~  $L_2$  norm of the residual (R) in the non-linear equations decreases by four orders of magnitude for the normal flow rule compared with two orders of magnitude for the non-normal flow rule for the same number of non-linear iterations (15 000); specifically to  $R = 8 \times 10^{-4}$  for  $e_G = e_F = 2$  to  $R = 6 \times 10^{-2}$  for  $e_G = 1.4$  and  $e_F = 2$ . Note that a Jacobian-free Newton-Krylov (JFNK) solver with a quadratic local  
355 numerical convergence does not perform better because the global convergence is poor with a combination of localized forcing and high grid resolution (Losch et al., 2014; Williams et al., 2017).
2. The width and activity of the LKFs is also affected by the flow rule. With  $e_G = 1.4$ , the lines are thinner, the shear along the LKFs is smaller and there is little shear between the fracture lines. With  $e_G = 4.0$ , the fracture lines are broader, the shear strain rate along the LKFs is higher and there is more shear between the fracture lines. With  $e_G = 1.4$  the ~~flow-rule~~  
360 deformation at the fracture is mainly in divergence, while for  $e_G = 4.0$ , the ~~flow-rule~~ deformation is mainly in shear and there is more stress transmitted to the ice in between the fracture lines.
3. The fracture angle changes as the plastic potential changes. The angles are wider with  $e_G = 4$  than  $e_G = 1.4$ . The effect of flow rule orientation on the fracture angles is discussed below.

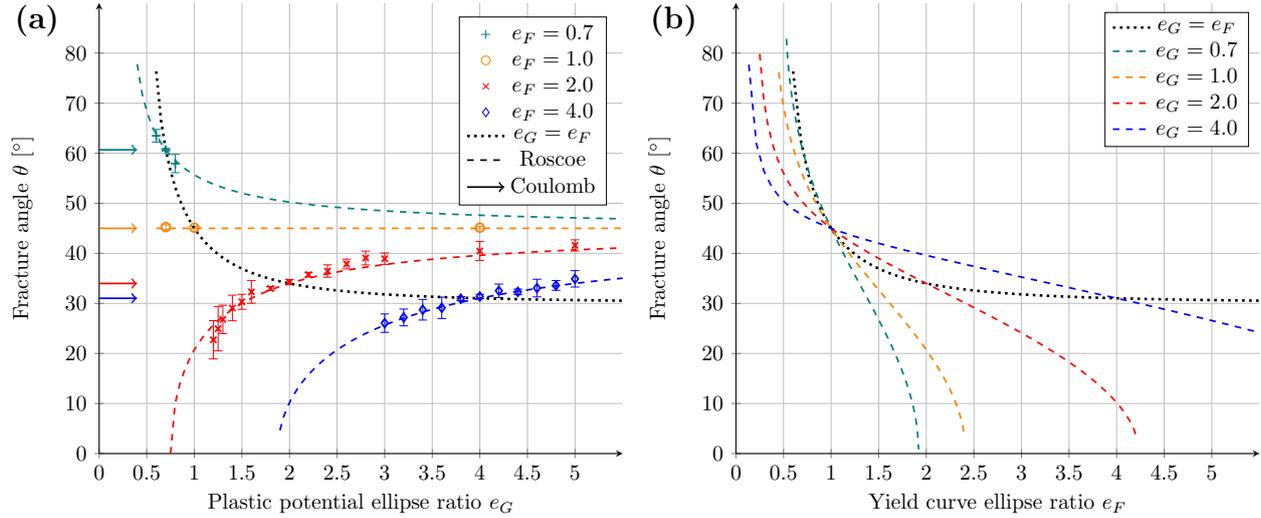
We now present results from four sets of simulations with fixed yield curve ellipse ratios at  $e_F = 0.7, 1.0, 2.0, 4.0$ . For each  
365 of these, we test the sensitivity of the results to changes in the plastic potential ellipse ratio  $e_G$ . The choice of yield curve ellipse ratios  $e_F$  are: the standard value of (Hibler, 1979), values suggested by Bouchat and Tremblay (2017) and Dumont et al. (2009), and an extreme value resulting in a very small shear strength and smaller fracture angles.



**Figure 6.** Diamond-shaped fracture pattern [in the shear deformation field  \$\dot{\epsilon}\_{II}\$](#)  for  $e_F = 2.0$  and three different values of  $e_G$  after five seconds of simulation. For the non-normal flow rule (panels a and c), there are primary and secondary fracture lines, in contrast to the normal flow rule (panel b) where single pair of fracture lines are simulated. The fracture angles are  $29.92 \pm 1.28^\circ$  for  $e_G = 1.4$ ,  $34.3 \pm 0.25^\circ$  for  $e_G = 2.0$ , and  $40.7 \pm 0.94^\circ$  for  $e_G = 4.0$ . The error corresponds to two standard deviation ( $2\sigma$ ) of the measured fracture angles.

Figure 7a shows how the fracture angles evolve as the plastic potential ellipse ratio  $e_G$  changes for each of the four values of  $e_F$ . There is a clear dependence of the fracture angles on the relative eccentricity of the plastic potential and yield curve. For  $e_G > e_F$ , the shear strain rate increases along the LKFs (see Fig. 6c) and the fracture ~~angle-angles~~ tend toward  $45^\circ$  as  $e_G$  increases, in agreement with the theory (Eq. 18). For  $e_G < e_F$ , the flow rule implies more divergence (for  $e_F > 1$ , or convergence for  $e_F < 1$ ) and less shear along the LKFs (see Fig. 6a), and the fracture angles move away from  $45^\circ$  as  $e_G$  decreases. More generally, for  $e_F < 1$ , the fracture angle increases with increasing convergence along the LKFs as  $e_G$  decreases. For  $e_F > 1$ , the fracture angle decreases with increasing divergence as  $e_G$  ~~becomes smaller~~ decreases. For  $e_F = 1$  (a circular yield curve), the fracture angles are independent of  $e_G$  because the fracture takes place at the peak of the yield curve and the flow rule is not affected by changes of the plastic potential ellipse ratio ( $e_G$ ).

The coloured dashed lines in Fig. 7a show the fracture angles  $\theta_{e,nn}(e_F, e_G)$  predicted by Eq. (30). The coefficient of determination  $R^2$  and the root-mean-square error (RMSE) between the simulated angle of fracture and theoretical predictions are  $0.974$   $0.97$  and  $0.37^\circ$  for  $e_F = 0.7$ ,  $0.953$   $0.95$  and  $1.22^\circ$  for  $e_F = 2.0$ , and  $0.968$   $0.97$  and  $0.47^\circ$  for  $e_F = 4.0$ . The RMSE is  $0.37^\circ$  for  $e_F = 1.0$ ,  $R^2$  being inapplicable. That is, the theory predicts the fracture angles accurately. This result shows that the flow rule plays a major ~~part~~ role in the simulated fracture angle for a given rheology. The black dashed line show the ~~evolution~~ ~~change~~ of the fracture angle with a normal flow rule ( $e_G = e_F$ , Eq. (31)).



**Figure 7.** **a)** Fracture angles as a function of the plastic potential ellipse ratio  $e_G$  for different yield curve ellipse ratios ( $e_F = 0.7, 1.0, 2.0,$  and  $4.0$ ). The markers with ranges are the mean and variation ( $2\sigma$ ) two standard deviations of the fracture angles. The dashed lines show the prediction from the theory Roscoe angle (Eq. 30). The arrows mark the Coulomb angles as a function of  $e_F$ , which are constant with respect to  $e_G$ . Colors indicate the value of  $e_F$  for lines and markers. The  $r^2$  between theory and modeled angles for  $e_F = 0.7, 2.0,$  and  $4.0$  are 0.97, 0.95, and 0.97. **b)** Theoretical predictions of the Roscoe fracture angle computed from Eq. (30) as a function of  $e_F$  with a constant  $e_G$ , for illustration. As  $e_F$  changes, both the stress state and the flow rule change (see Fig. 4), resulting in a more complex behavior. The black dotted line for the normal flow rule ( $e_F = e_G$ ) is indicated drawn for reference.

For completeness, Figure 7b also show the theoretical predictions for a constant plastic potential ellipse ratio  $e_G$  as the for varying yield curve ellipse ratio ratios  $e_F$  changes. The modeled angles for  $e_G = 4.0$  are shown as an example. The fracture angles become smaller as the  $e_F$  increase. Yield curve ellipse ratio smaller than  $e_F = 1$  do not create fracture angles below  $45^\circ$ .

## 5 Discussion

The idealized experiments using the elliptical yield curve with a non-normal flow rule show confirm that the type of deformation and the fracture angle are intimately linked with the shape of the plastic potential. We observe that, irrespective of the plastic potential elliptical aspect ratio, a yield curve ellipse ratio  $e_F < 1$  does not allow for fracture angles smaller than  $45^\circ$  in uniaxial compression when using an elliptical plastic potential, irrespective of the plastic potential elliptical aspect ratio. To reduce the fracture angles with yield curve ellipse ratios  $e_F > 1$ , one needs to use plastic potential ellipse ratios  $e_G$  smaller than the yield curve ellipse ratio are required, that is,  $e_G < e_F$ . The numerical idealized experiments show that the use of with a plastic potential in a viscous-plastic model allows separating we can separate the yield criterion from the resulting deformation (flow rule). This allows to decouple decoupling the mechanical strength properties of the material (ice) from its post-fracture

~~behaviour~~behavior. The results illustrate clearly how the yield curve defines the stress for which the ice will deform, that is, the transition between viscous and plastic deformation, and how the relative shape of the plastic potential with respect to the yield curve defines both the fracture angle and the type of deformation (convergence or shear) along the fracture line ~~and the fracture angle~~. The resulting fracture angles are in excellent agreement with the Roscoe angle predictions (Roscoe, 1970).

400 Understanding the link between rheology and fracture angle is necessary for choosing or designing a rheology that is capable of reproducing the observed ~~angle of intersection between~~ intersection angles between pairs of LKFs and consequently the emerging anisotropy. ~~In principle, it may be possible to solve~~ A independent plastic potential may resolve several inconsistencies of the standard elliptical yield curve with a normal flow rule ~~Ringeisen et al. (discussed in 2019) by using a plastic potential that is independent of the yield curve~~ (discussed in Ringeisen et al., 2019), namely:

405 1. In the standard VP model with an elliptical yield curve and normal flow rule, adding shear strength increases the fracture angle, ~~contrary in contradiction~~ to granular matter theory (Coulomb, 1773). This ~~behaviour~~behavior is linked to the specific shape of the elliptical yield curve with a maximum shear stress at  $P/2$  and an ascending and a descending part. In principle, we can decrease the fracture angle with increasing shear strength ( $e_F$  decreasing) by decreasing  $e_G$ , but only if  $e_F > 1$ . ~~When doing this,~~ but then the flow rule ~~becomes very non-normal~~ is far from “normal”, making the

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numerical convergence difficult.

2. ~~The~~ Because of the elliptical shape of the yield curve, the angle of fracture in the standard VP model changes with confining pressure (Ringeisen et al., 2019, Sec. 3.2.2, Fig. 8) unlike laboratory experiments with granular materials (e.g., sand) where the fracture angle is ~~relatively insensitive~~ only weakly sensitive to the confining pressure (~~Alshibli and Sture, 2000~~) ~~–This behaviour~~ (Han and Drescher, 1993; Desrues and Hammad, 1989; Alshibli and Sture, 2000). This ~~behavior~~ cannot be eliminated with an elliptical plastic potential, as the normal ~~stresses~~ stress along the LKFs increases with confining pressure and the flow rule changes from divergence to convergence as one passes the maximum shear stress at  $P/2$  (Ringeisen et al., 2019). A different plastic potential function would change this ~~behaviour~~behavior. However, this would make the model implementation and numerical convergence even more difficult. ~~However, we~~ We note that a 3D granular material like sand cannot release stress by ridging as sea ice does. A 2D material, such as sea ice, can

415

420 ridge and “escape to the 3rd dimension” after fracture. Therefore, we expect a change in the fracture angles at large confinement. Laboratory experiments show this behavior and yield stresses in sea ice change above a critical confinement ratio (Golding et al., 2010; Schulson, 2002). It is still not clear whether these results can be extrapolated to the modeling sea ice as a 2D medium at the geophysical scale, although several common features can be found (Schulson, 2002).

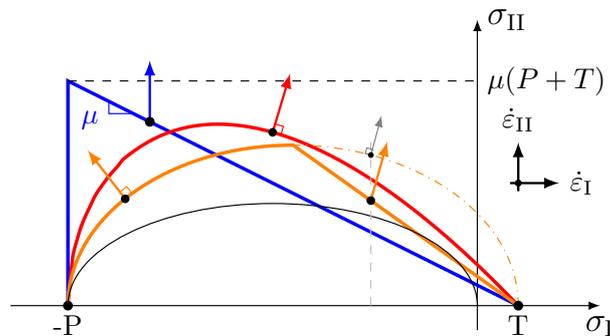
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3. In the standard VP model with a normal flow rule, the divergence and convergence are set by the ellipse ratio of the yield curve, and thus by the relative amounts of compressive and shear stress. The plastic potential ellipse ratio  $e_G$  changes the flow rule but does not change the sign of the divergence along the LKFs which is solely determined by the yield curve ellipse ratio  $e_F$ . With the elliptical plastic potential, convergent motion remains convergent and only the ratio of shear to convergence changes. To change this ~~behaviour~~behavior, a different shape of plastic potential is required, for example a teardrop plastic potential.

430 4. The fracture angles in the standard VP models are larger than observed. Using a non-normal flow rule allows us to change the fracture angle in uni-axial compression to values below  $30^\circ$ ; ~~something that~~. This is not possible with a normal flow rule (Ringeisen et al., 2019).

### Other yield curves

435 We discuss the elliptical yield curve here because it the most commonly used one and its behavior is better documented than any other model in use in the community. This provides a known reference for studying the use of non-associated flow rules. Our goal is to provide a reference for the future development of viscous-plastic rheologies with non-normal flow rules rather than suggest a new VP rheology. Alternatives to the elliptical yield curve have been used in previous studies before; for instance, the Mohr–Coulomb, the Coulombic yield curve, or the teardrop yield curves ~~–The use~~ (Figure 8). The concept of a plastic potential in conjunction with these yield curves may also prove useful in solving ~~these issues~~ the issues described above.  
 440 A detailed analysis of the simulations using the family of Mohr–Coulomb and Teardrop yield curves is beyond the scope of this work and will be presented in a subsequent study. Below, we ~~apply knowledge gained from the simulations presented above, and discuss~~ use the experience from our simulations to infer how alternative yield curves may address deficiencies in the standard VP rheology.



**Figure 8.** Alternative yield curves and flow rules: The Mohr–Coulomb yield curve with shear (a non-normal) flow rule (blue, Ip et al., 1991), the modified Coulombic yield curve with normal (elliptic part) and non-normal (linear part) flow rule (orange, Hibler and Schulson, 2000), and the teardrop yield curve with a normal flow rule (red, Zhang and Rothrock, 2005). The elliptical yield curve with  $e_F = 2.0$  is shown for reference (black thin line).  $P$  is the compressive ice strength, and  $T$  the tensile ice strength.

445 ~~It is possible to include different (non-normal) flow rules~~ Non-normal flow rules can be combined with the Mohr–Coulomb family of yield curve. ~~The Mohr–Coulomb~~ For a Mohr–Coulomb yield curve with a ~~pure shear flow rule (Ip et al., 1991)~~ would create fracture angle approximately equal to double sliding law (i.e., pure shear deformation, Ip et al., 1991), the Roscoe theory predicts a fracture angle of approximately  $45^\circ$ , independently that is independent of the slope of the yield curve. ~~It corresponds to the case~~ This behavior can be mimicked using an elliptical yield curve and plastic potential by setting  $e_G \gg e_F$ , hence  $\delta \simeq 0$  and  $\theta = 45^\circ$ , as shown in (Fig. 7a). This contradicts the Coulomb ~~angle~~ theory, which predicts an angle of fracture that depends ~~solely exclusively~~ solely exclusively on the internal angle of friction (Eq. 1). ~~Including~~ Combining an angle of dilatancy with  
 450

a Mohr–Coulomb yield curve (Tremblay and Mysak, 1997) would allow ~~for different angles of fracture~~ an angle of fracture depending on  $\delta$  that is different with shear and divergence ~~or convergence ( $\delta > 0$ ) or with convergence ( $\delta < 0$ )~~ along the LKFs ~~depending on  $\delta$ . Such. Such a~~ fracture angle and divergence would be independent of the shear strength and the confining pressure in agreement with Roscoe’s angle of fracture ~~–Such so that such~~ a rheology could potentially solve ~~the~~ all four  
455 issues on page 18. It is also important to note that the Mohr–Coulomb yield curves do not satisfy the convexity requirements of Drucker’s ~~postulate of stability~~ stability postulate. Mohr–Coulomb yield curves in plastic ~~earth mantle models showed different~~ Earth mantle models lead to a variety of fracture angles corresponding to the Coulomb angle, Roscoe angle, and the intermediate Arthur angles (~~Buiter et al., 2006; Kaus, 2010; Mancktelow, 2006~~) (Buiter et al., 2006; Kaus, 2010; Mancktelow, 2006). However, such geological models are ~~usually incompressible~~ designed for incompressible medium, and making inferences for  
460 the compressible formulation of sea ice models is difficult. ~~The investigation of the fracture angles with Mohr–Coulomb yield curves is left for future work.~~

The Coulombic yield curve uses the two straight limbs from the Mohr–Coulomb yield curve and an elliptical cap of the standard VP rheology for large compressive stresses (Hibler and Schulson, 2000). In this rheology, the flow rule over the two straight limbs is defined by the elliptical yield curve; that is, the ellipse serves as a plastic potential for the ~~Mohr–Coulomb~~  
465 Mohr–Coulomb yield curve. The Coulombic yield curve leads to unrealistic and asymmetrical fracture lines (*i*) when the stress states ~~lie at fall onto~~ the non-differentiable intersection between the straight limbs and the elliptical cap (Ringeisen et al., 2019), and (*ii*) when the stress states ~~lie on fall onto~~ the two straight limbs with the non-normal flow rule. Note that straight and symmetric fracture lines in this rheology are only possible when all the stress states are ~~located~~ on the Mohr–Coulomb limbs and the flow rule at the fracture line is near-normal, that is, at the location where the normal to the ~~ellipse~~ elliptic plastic potential  
470 is nearly perpendicular to the limbs of the Mohr–Coulomb yield curve (Ringeisen et al., 2019). Hibler and Schulson (2000) already inferred that the flow rule may have an effect on the angle of fracture, but the authors limited their case to the framework of flawed ice and did not consider Roscoe’s theory of dilatancy. The rheology of Hibler and Schulson (2000) was tested in an idealized experiment more complex than ours (Hutchings et al., 2005), but the effect of using a non-normal flow rule was not explored. The complexity of their setup may explain the observed difference between simulated and predicted angles. Note that  
475 the rheology in Hibler and Schulson (2000) was built by changing the shape of the yield curve a-posteriori, while the rheology presented here solves the constitutive equations rigorously.

The teardrop yield curve with a normal flow rule (Rothrock, 1975; Zhang and Rothrock, 2005) is divergent for a wide range of normal stresses and for all practical purposes consists of a continuously differentiable version of the Coulombic yield curve. This asymmetry between divergent and convergent deformation for different normal stresses decreases the effect  
480 of confinement on the fracture angle ~~—issue—~~ issue 2 ~~—on page 18—~~ and reduces the fracture angle for any confinement pressure ~~—issue—~~ issue 4. This yield curve does not address issue 1, because adding shear strength in the ~~tear-drop~~ teardrop yield curve also increases the fracture angle.

~~The~~ As the main disadvantage of a non-normal flow rule ~~is the slower numerical convergence~~ we found that it leads to slower convergence of the numerical solver. Solving the momentum equation accurately requires more solver iterations and  
485 failure to converge is more frequent than for standard normal-flow-rule rheologies. In our simulations, this numerical issue

manifests itself by the presence of multiple and asymmetrical fracture lines despite the fact that our ~~experiments are entirely symmetrical.~~ experiment geometry and forcing are exactly symmetrical. This asymmetry is not expected, and is not found with normal flow rules. The fracture lines with a normal flow rule are symmetrical and come in pairs (Ringey et al., 2019). ~~The poorer numerical convergence in practice~~ In practice, the numerical convergence issue will go unnoticed in high-resolution 490 simulations using realistic geometries, ~~since and time varying wind forcing.~~ In these simulations, while the number of iterations typically used ( ~~$O(10)$~~  $O(10)$ ) is much smaller than that required for full convergence. ~~On the upside,~~ at each time-step, a new iteration typically ~~use the solution from~~ uses the solution of the previous timestep as ~~initial conditions.~~ This ~~the initial estimate.~~ With this, together with slowly varying forcing in space and time, the number of solver iterations per forcing cycle is large, in contrast to the fast changing forcing in this study (every timesteps). Whether this ~~behaviour~~ behavior (asymmetry and multiple 495 fracture lines) will also be present in realistic simulation using spatially and temporally varying wind forcing remains to be tested.

The following criteria should be considered when building a new rheology: The spatial and temporal scaling of sea-ice deformation should agree with observations (Bouchat and Tremblay, 2017; Hutter et al., 2018); the flow rule should reproduce the correct divergence along LKFs (Stern et al., 1995); the yield curve ~~includes~~ should include some tensile strength 500 (Coon et al., 2007) and ~~reproduces observed distributions of internal stress when ice deforms~~ be Coulombic in nature in agreement with observed internal stress invariants from ice stress buoys (Weiss and Schulson, 2009); the distribution of fracture angles should agree with observations (Marko and Thomson, 1977; Erlingsson, 1988; Cunningham et al., 1994; Hutter et al., 2019); the sea ice mechanical strength properties (i.e., yield curve) and deformation (~~flow rule~~ i.e., flow rule for VP rheologies) should vary in time and space depending on, for example, the time-varying distribution of the contact 505 normals ~~and,~~ floe size distributions, or a damage parameter, as per observations and laboratory or numerical experiments (~~Overland et al., 1998; Hutter et al., 2019; Horvat and Tziperman, 2017; Roach et al., 2018; Balendran and Nemat-Nasser, 1993; Dansereau et al., 2019~~ Overland et al., 1998; Hutter et al., 2019; Horvat and Tziperman, 2017; Roach et al., 2018; Balendran and Nemat-Nasser, 1993; Dansereau et al., 2019).

Although high spatial resolution observations from satellite are available from optical instruments (e.g., from the Landsat 510 or Sentinel programs), higher temporal resolution of sea ice deformation and flow size distributions ~~are still lacking.~~ So is the combined knowledge of the failure stresses and their associated deformation of sea ice as is still unavailable. The new Sentinel constellation and in-situ observations from the field program MOSAIC may bridge this gap. There is also a knowledge gap in the interplay between yield stresses and the post-fracture deformation in a 2D granular material. granular material such as sea ice. This interplay is likely different than for the well studied case of a solid homogeneous 3D block of ice (e.g. Schulson, 2002) 515 Sea ice floating on the ocean surface can “escape” vertically when ridging it forms ridges under confined compression (Hopkins, 1994). This ~~behaviour~~ behavior differs from laboratory test ~~that use axial symmetry and general knowledge about with~~ 3D granular material like sand that use axial symmetry. Generally, information about sea ice resistance in different configurations (e.g., confinement) and the resulting fracture angles and deformation (ridging or opening) is also still missing, although some laboratory scale experimental results are available Weiss et al. (2007); Schulson et al. (2006); Weiss and Schulson (2009). 520 The sea ice flow size distribution varies in time (summer/winter) and space (marginal ice zone/central Arctic) (Rothrock and

Thorndike, 1984). These variations change the mechanical properties (e.g., distribution of contact normals) and thermodynamic properties (e.g., lateral melt) of sea ice (Horvat and Tziperman, 2017). ~~Consolidated~~ Designing more appropriate rheologies for improved high-resolution climate models and more accurate sea ice prediction systems requires consolidated observations of these ~~two physical processes are needed to design new rheologies for high-resolution climate modelling, still unclear physical~~ processes.

## 6 Conclusions

The flow rule, which dictates the post-fracture deformation, has a fundamental effect on the orientation of fractures lines in a viscous-plastic sea ice model. To test this, we added an elliptical plastic potential (allowing for a non-normal flow rule) to the standard viscous plastic rheology with an elliptical yield curve, therefore modifying the flow rule without changing the yielding stress state. We tested this new rheology with numerical experiments in uni-axial compression using the standard viscous-plastic model of Hibler (1979). The modeled fracture angles are in agreement with the Roscoe angle, a theory based on experiments with granular materials that includes an angle of dilatancy (Roscoe, 1970; Tremblay and Mysak, 1997). This new rheology partially solves issues raised in an earlier study (Ringeyisen et al., 2019). The use of a plastic potential or non-normal flow rule allows for the ~~simulations~~ simulation of smaller fracture angles between pairs of Linear Kinematic Features, in agreement with satellite observations. ~~The fracture angles, however, Because of the elliptical yield curve, the fracture angles~~ still depend on the confinement pressure (Ringeyisen et al., 2019), and the elliptical plastic potential does not modify only modifies the ratio of divergence relative to shear, but not the direction of deformation at the fracture lines (convergence or divergence), ~~only the ratio of divergence relative to shear.~~ The momentum equations for a rheological model with a non-normal flow rule are more difficult to solve numerically, and produce multiple lines of fractures that are asymmetrical (despite the geometrical symmetry of the problem), in contrast with a model with a normal flow rule. ~~Understanding~~ It is necessary to understand the effect of the flow rule on the fracture angle ~~is necessary~~ to design VP rheologies for high-resolution sea-ice modeling that both reproduce fracture angles and deformation along the fracture lines ~~, and the behaviour and the behavior~~ of sea ice as a granular material.

Designing a rheology for high-resolution ~~simulation requires information on~~ simulations requires information about sea ice fracture angles and sea ice strength in a wide range of stress conditions (i.e., compression with or without confinement, pure shear, tension), yet unavailable at high temporal and spatial resolution. The observations of the Multidisciplinary drifting Observatory for the Study of Arctic Climate (MOSAIC, Dethloff et al., 2016) in 2019/2020 may provide valuable data from continuous ice radar imaging, stress sensors, and arrays of drift buoys that will greatly help improve sea ice model dynamics.

*Code availability.* The modified version of MITgcm used in this study is available at [https://github.com/dringeis/MITgcm/tree/ell\\_nnfr](https://github.com/dringeis/MITgcm/tree/ell_nnfr)

550 *Author contributions.* DR designed the rheology and implemented the code changes with ML. DR ran the experiments. DR and BT designed the theory linking sea ice rheologies and granular matter theory. DR wrote the manuscript with contributions of BT and ML.

*Competing interests.* The authors declare that they have no conflict of interest.

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