Responses to Referee comments by Anonymous Referee #1 on "Brief communication: Time step dependence (and fixes) in Stokes simulations of calving ice shelves" by Brandon Berg and Jeremy Bassis

We thank the anonymous reviewer for their feedback on this manuscript. Our responses to comments are given below, with original comments in black and responses in red. When referenced, line numbers refer to the revised manuscript.

General Comments:

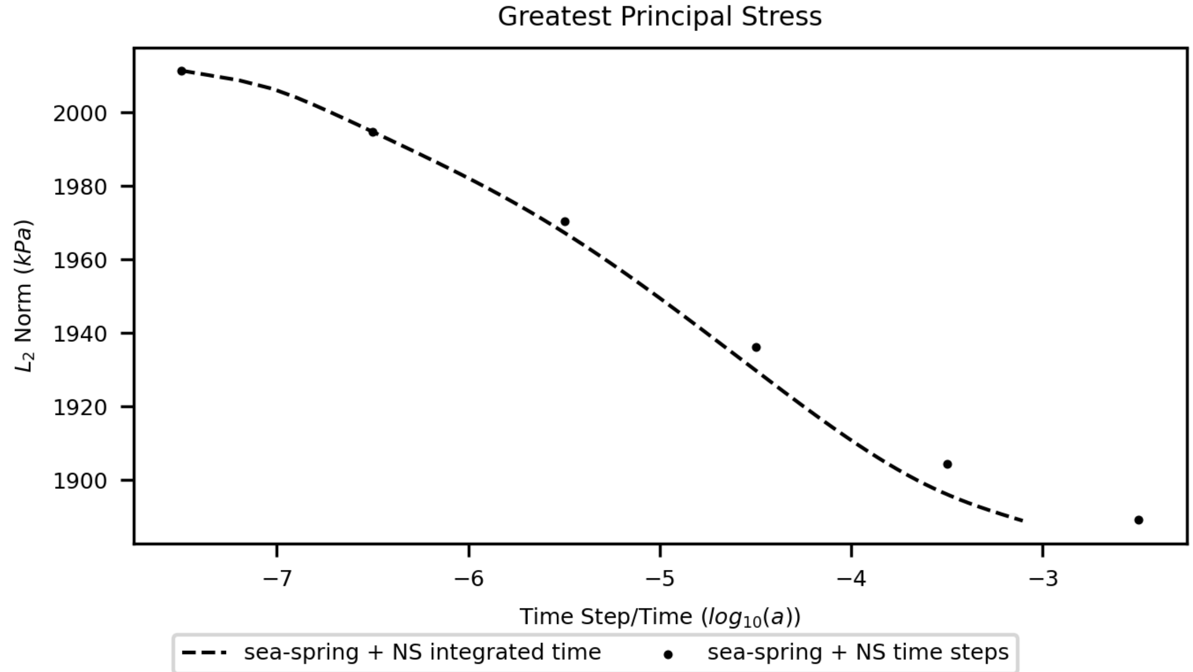
This paper presents a simple method to overcome time step dependence of the solution arising when solving for an ice-shelf which departs significantly for hydrostatic equilibrium. This could be the case for instantaneous non-vertical icebergs calving or supraglacial lake drainage. This is quite a technical paper but as the problem might be encountered by other groups using different Stokes solvers, this brief communication certainly deserves to be published. The overall writing of the paper is quite good even if I think that there is some room for improvement.

We thank the reviewer for their positive comments. We respond in more detail to each comment below.

My main concern is the fact that the time step dependence of the solution is sometimes seen as negative (e.g. title) or positive (e.g. caption Fig. 2). And indeed it is not completely clear from Figs. 2 or 3 to see which of the two solutions is the one that works better.

This is a good point and we agree that the original figures were confusing. We have added text to the Figure 2 caption clarifying that the time step variability for the sea-spring with Navier Stokes solution is connected to the time evolution of the *system*. In the classic Stokes system, the velocity depends solely on the geometry of the ice shelf/glacier and internal properties (e.g., temperature). Hence, the dependence of the velocity field on time step is unphysical. However, when solving the full Navier-Stokes system, the velocity becomes time dependent and, like all numerical ODE integrations, we must take sufficiently small time steps to ensure numerical convergence when integrating with respect to time to find the numerical approximation to the solution. For the sea-spring + NS solution, the velocity tends towards zero as time step size decreases. As a consequence there is no deformation and, for very small times, the ice shelf behaves as a nearly rigid body as it approaches hydrostatic equilibrium. In the sea-spring + NS method, taking small time steps allows us to resolve the quasi-rigid body uplift of the ice shelf as it "bobs" in the water. We have added text in Section 4 of the manuscript clarifying the rigid body behavior at short time steps (lines 132-135).

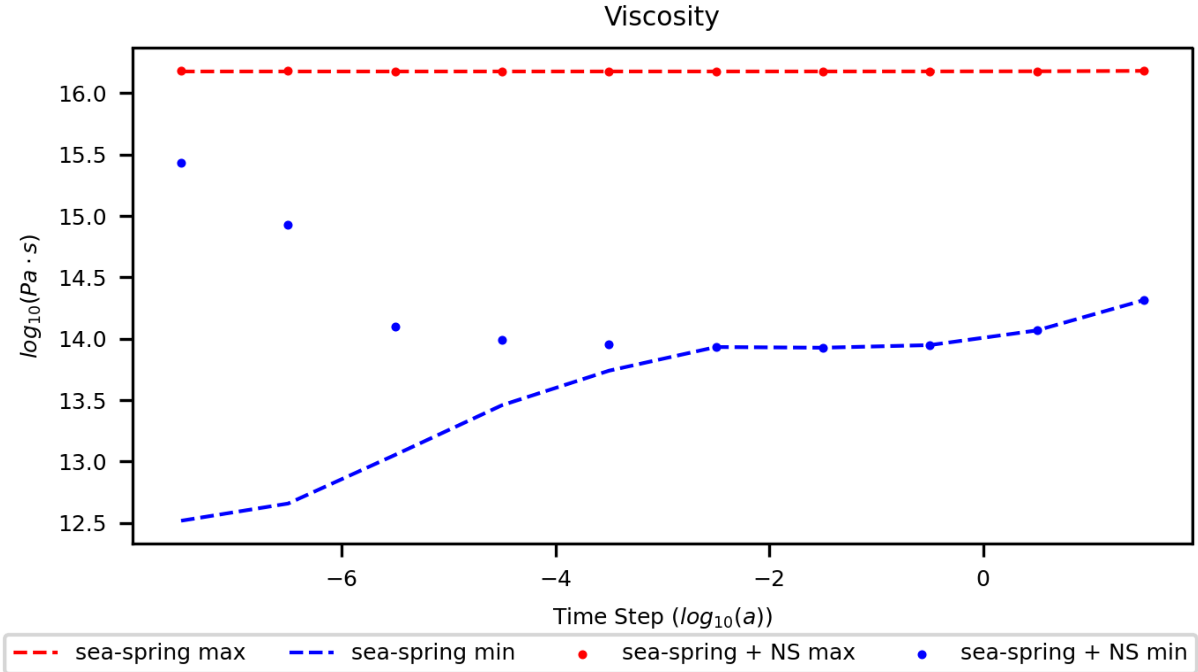
This is illustrated below, where we show a plot of the L2 norm of the greatest principal stress for the sea-spring + NS method. The points represent different time step sizes (as in the manuscript). The dashed line is created by choosing the smallest time step and numerically integrating the system in time. For small times, the solution obtained from taking a single step with different time step size and the solution obtained from numerical integration (with a very small time step size) are similar, but begin to differ as time step size increases because taking large time steps results in less accurate solutions. Thus, the variation from the sea-spring + NS solution at small time steps is consistent with the actual time evolution of the system. This is connected to our discussion in section 4 of the manuscript.



The viscosity has no timestep dependence for the sea-spring solution and it is the sea-spring+NS solution that has no time step dependence for effective strain-rate.

We have chosen to omit plotting the viscosity form the final manuscript, instead replacing it with a plot of vertical velocity because effective strain rate and viscosity display somewhat redundant information and to ease the exposition of this short manuscript. Showing the vertical velocity directly provides a more direct illustration of the problem with the vertical velocity becoming unphysically large as time step size decreases.

However, we include a plot of viscosity below that shows the minimum and maximum value of the viscosity at different time steps for the two methods. Plotting the maximum and minimum values better highlights the time step dependence of the viscosity at small time steps. Note that for small time step sizes (and times) with the sea-spring + NS solution, the motion is quasi-rigid (i.e., deformation is small), strain rates are small and the viscosity increases. The viscosity is ultimately limited by the regularization we use for the rheology at small strain rates. By contrast, for the sea-spring solution, as time step size decreases the increasingly large vertical velocity causes a large strain rate increase. Because the viscosity and strain rate are inversely related, this results in decreasing viscosities.



This is even less clear for stress where both solutions are diverging but presents both a timestep dependence. I would expect more comments on the text on this and how from the figure one can decide which is the working solution.

We have clarified our language and removed most uses of the word "divergent" from the manuscript. The vertical velocity, which we now show in Figure 2 panel (a), becomes unphysically large for small time step sizes for the sea-spring method. Effective strain rate becomes approximately zero for small time step sizes for the sea-spring + NS method because the motion at small times is nearly rigid body. Effective shear stress tends to zero for the sea-spring + NS method for the same reason as the effective strain rate. We also point out in section 4 that while Figure 3 shows higher maximum stresses for the sea-spring method, the L2 norm shows higher stresses for the sea-spring + NS method because of high negative compressive stresses (lines 136-137).

Smaller Points:

page 2, line 41: ", where u1 is a constant"

Comma has been added.

Figures 2 and 3: the quality of Figs. 2 and 3 are very low.

Figures have been changed from .png to .eps to improve viewing quality.

It is not clear from the text and the captions if what is plotted on these figures is the solution after the timestep following the calving event.

Thanks for pointing this out. We now state in the figure captions that the plotted solution is immediately after the (emulated) calving event.

What are the differences of setup between Fig. 2 c and d and Fig. 3?

The difference between Fig. 2 c and d and Fig. 3 is that we plot the L2 norm of the solution in Figure 2 and the L1 norm (maximum) of the solution in Figure 3. This is emphasized in the text and on figure y-axes. We plot the L1 norm (maximum) because this is the criterion that is often used in stress based calving laws, like the Nye zero stress. We show the L2 norm because it is (often) a more robust diagnostic of the behavior of the numerical solution.

I would suggest to modify Pa to MPa or kPa. For the x axis, the caption should tell that time step are varying from xx seconds to xx years?

We have modified the axes for stress to be kPa rather than Pa. Figure caption now states that the time step ranges from 1 second to 30 years.

page 5, line 95: not sure the second sentence of part 3.2 should start with "Furthermore"?

Thanks. “Furthermore” has been removed.

Eq. (9): specify that ui−1 is the velocity at previous timestep?

A sentence has been added after the equation defining the variable.

page 6, line 113: "where the damping coefficient is"

Correction has been made.

Responses to Referee comments by Christian Schoof on “Brief communication: Time step dependence (and fixes) in Stokes simulations of calving ice shelves” by Brandon Berg and Jeremy Bassis

We thank Christian Schoof for his feedback on this manuscript. Our responses to comments are given below, with original recommendations/points in black and responses in red. When referenced, line numbers refer to the revised manuscript.

Major Recommendations:

We thank the reviewer for his comments and suggestions. We have incorporated most of the reviewers' excellent suggestions in the manuscript. We have vacillated slightly in our preferred terminology between ill-posed and stiff before settling on “unphysical” for reasons that are described in more detail in response to specific reviewer comments.

I would make clear the distinction between the poorly condition Stokes flow problem in Durand et al and the two flavours of actual ill-posedness seen when your boundary conditions permit force and/or torque balance to fail. It won’t hurt to allude to the latter, even though I don’t imagine many people are trying to solve Stokes flow problems for icebergs — you never know. It would also be reasonable to say that the sea spring mechanism (probably) works well for the second version of the actually ill-posed case, where big departures from equilibrium need never occur.

This is a good point. Our emphasis here was really on pointing out that when the geometry evolves rapidly, as is the case for an iceberg calving event, the sea-spring method becomes problematic and can lead to numerical problems. And when these numerical inaccuracies are combined with, for example, stress based calving criteria, there is the possibility of introducing purely numerical calving instabilities. However, under most circumstances, the sea-spring method remains satisfactory. This is better emphasized in lines 69-76.

We have added additional text to clarify the fact that, for our choice of boundary conditions, global force and torque balance are not necessarily satisfied leading to an ill-posed problem (lines 54-56, lines 110-114). However, if we consider fixed velocity (Dirichlet) boundary conditions over a portion of the domain, there is no rigid body motion (translation or rotation) that can be added to the ice shelf. In this case we can still obtain large velocities that are time step dependent using the sea-spring method when the geometry departs significantly from hydrostatic equilibrium over a portion of the domain.

Starting with a geometry that exactly satisfies global and local force/torque balance and then introducing small changes to that geometry can result in large changes to the velocity. This is what we think the reviewer calls “stiff” or poorly conditioned. Small changes in the initial conditions (i.e. position of the ice water interface) lead to large changes in the velocity solution. This can be partly cured by adding, say, a quadratic drag force due to the water, as the reviewer notes. However, for realistic drag coefficients, this still results in exceptionally large velocities. In fact, for configurations that we tested, the velocities can exceed the speed of sound! Because of this and because of the fact that we wish to avoid any confusion between “ill-posed”, “ill-conditioned”, and “stiff”, we have decided to rephrase and call this behavior “unphysical”. We believe this captures the numerical issue accurately and avoids introducing additional jargon that glaciologists might not be as familiar with.

The paper says ‘However, as we shall show, this assumption is problematic for applications where the ice departs from hydrostatic equilibrium.’ I’d circle back to this at some point and point out that things may not be quite so dramatic as to say that the Stokes equations have nothing to say about what happens during these ‘inertial’ events; they do, but in modified form. I think this will also tie in to the discussion of how to formulate the inertial term in discrete form, see again under ‘minor points’ below.

As stated above, if we consider fixed velocity (Dirichlet) boundary conditions over a portion of the domain, there is no rigid body motion (translation or rotation) that can be added to the ice shelf and we still obtain large velocities that are time step dependent using the sea-spring method. The large velocities are tied to the bending of the ice that occurs in response to removal of ice at the calving front. As the reviewer notes, this is because the problem is “stiff”.

But to simplify our discussion, we have removed equation (8) and the accompanying text regarding the separation of the velocity into viscous and uplift components. Instead, we focus on highlighting the “stiffness” of the problem and how small changes to the ice-ocean boundary location can cause large changes in the solution. In this way, we emphasize the importance of carefully treating the hydrostatic uplift without commenting on the exact nature of the decomposition of viscous and rigid body motion. However, we do add text in the manuscript stating that such a decomposition may be possible (lines 110-114).

Minor Points:

I would probably make a bit clearer how boundary conditions are important in determining whether an actual ill-posedness can occur in the sense of there being no solution to the Stokes flow problem. As the extended discussion above indicates, the partial Dirichlet conditions in the present paper ensure there is no issue of torque or horizontal force imbalance, but it others may run into these issues in their own research, and look to apply the method developed here. Also, as indicated in the second paragraph of this review, there may be other tricks to ensuring force balance.

We have added text to clarify this. In particular, we have noted that adding global constraints on force and torque balance is possible (lines 110-114). We do, however, note that because ice breaks, global force and torque balance would have to be considered on each intact segment and this becomes increasingly challenging to efficiently identify and manage. Hence, our solution of simply including the acceleration directly into the Stokes equations becomes a more practical solution. As noted in our previous response, we have also shifted our terminology to “unphysical” because unphysically large velocities are still possible when drag is included.

The decomposition in equation (8): for a sea spring model, I don’t think you can argue that the sea spring term $u(x, z)\delta t$ simply causes an additive term $\Delta z_{uplift}/\delta t$…

We have streamlined and simplified this section and have eliminated this equation and explanation. We now focus on the “stiff” nature of the problem rather than a specific decomposition into viscous and uplift components.

Notation: there is quite a bit of randomness about which quantities are in boldface and which are not, especially when it comes to tensors (σ versus ε and I?). Make it consistent to please the eye. . .

Notation has been changed so that both vectors and tensors are all in boldface.

The '/2' should probably be inside the square root in the definition of the invariant εe just after equation (4)

Error has been fixed.

Writing $u(\Delta t)$ on the left-hand side of (8) is confusing as u has a well-defined meaning as the continuum solution of the Navier-Stokes problem as a function of (x, z, t), so changing the arguments of that function haphazardly to Δt is bad form (and actually confused me quite a bit). For starters, the quantity on the left isn't u but a numerical approximation to it, solving a modified problem, so give it a different symbol, and be clear why you are using the Δt argument on the left (your numerical algorithm thus constructed leads to a solution that turns out to depend on Δt, whereas you would want it not to be dependent on Δt.

We have eliminated Equation (8) in response to other comments by the reviewer.

The numerical form of the acceleration term in equation (9): this is defensible when you have no rotational degrees of freedom in the rigid body motion, because the advection term for momentum $u \cdot \nabla u$ is dominated by $r \cdot \nabla r$ (see point 2 above), and in the absence of a rotational degree of freedom, $\nabla r = 0$ so the advection term goes away. As soon as there is rotation, this is no longer true, and you are wel advised to retain the full inertial term $\partial u/\partial t + u \cdot \nabla u$. I realize that the present paper does not allow for that possibility, but I think it is worth mentioning.

In our model, we are using an Arbitrary Langrangian Eulerian (ALE) formulation, which updates all mesh coordinates at every time step based on the velocity field. This led us to use the material derivative in Equation (9). But we have added text to clarify the fact that even in a Eulerian reference frame we could neglect the $u \cdot \nabla u$ term because the velocity field does not contain a rigid body rotation (lines 117-118). We have also added text explicitly stating we are using an ALE formulation (lines 90-92).

Again, equation (9): I am actually not clear how you imagine you are computing this in a 'Lagrangian frame' to begin with, since you are solving, in discrete terms, an elliptic equation (or a parabolic equation with a backward Euler step, which is the same thing); if you genuinely are using a Lagrangian transformation here, please be explicit and speciifc. In terms of implementation, the reduced acceleration term in equation (9) is my biggest concern, even if I believe it to be leading-order correct (in the Reynolds number, see above) for the vertical-motion-only case discussed in the paper.

We are not quite sure that we understand the reviewer's question. In our implementation, we are solving a parabolic equation with a backward Euler step. The update to the ice geometry is done using a fully Langrangian formulation in which we update the mesh coordinates at every time step. Of course, we do need an initial condition for particle velocities. For the initial condition on velocity to use in the backward Euler step, we choose a uniform velocity field equal to the inflow velocity in the horizontal direction and zero in the vertical direction. The choice of zero initial velocity in the vertical direction is motivated by the experimental design, in which we imagine an ice shelf that is initially perfectly at hydrostatic equilibrium, and thus should have nearly zero vertical velocity before calving. We have added text to section 4 of the manuscript to clarify the precise initial condition on velocity (lines 118-121).

Brief Communication: Time step dependence (and fixes) in Stokes simulations of calving ice shelves

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Abstract. The buoyancy boundary condition applied to floating portions of ice sheets and glaciers in Stokes models ~~is numerically ill-posed~~ requires special consideration when the glacier rapidly departs from hydrostatic equilibrium. This ~~manifests in velocity solutions that diverge with decreasing~~ boundary condition can manifest in velocity fields that are unphysically (and strongly) dependent on time step size, ~~contaminating diagnostic strain rate and~~ thereby contaminating diagnostic stress fields. This can be especially problematic for models of calving glaciers, where rapid changes in geometry ~~lead to configurations that~~ cause configurations that suddenly depart from hydrostatic equilibrium and ~~accurate measures~~ lead to inaccurate estimates of the stress field~~are needed~~. Here we show that the ~~singular~~ unphysical behavior can be cured with minimal computational cost by reintroducing a regularization that corresponds to the acceleration term in the stress balance. This regularization provides ~~numerically stable~~ consistent velocity solutions for all time step sizes.

1 Introduction

Stokes simulations are used in glaciology as a tool to determine the time evolution of glaciers (e.g., Gagliardini et al., 2013). Increasingly, these models are also used to examine the stress field within glaciers to better understand factors that control crevasse formation and the onset of calving events (Ma et al., 2017; Benn et al., 2017; Nick et al., 2010; Todd and Christoffersen, 2014; Ma and Bassis, 2019). This type of model can provide insight into the relationship between calving, climate forcing, and boundary conditions (e.g., Todd and Christoffersen, 2014; Ma et al., 2017; Ma and Bassis, 2019).

Here we show that a common method used to implement the ice-ocean boundary condition in Stokes models can result in solutions that are unphysically sensitive to the choice of simulation time step size. This behavior manifests in applications that allow for rapid changes in the model domain — a type of change associated with models that allow for instantaneous calving events or crevasses (Todd and Christoffersen, 2014; Todd et al., 2018; Yu et al., 2017).

The time step dependence arises because for glaciers outside of hydrostatic equilibrium, the acceleration is not small ~~,~~ as assumed in Stokes flow. We illustrate both the issue and the solution using an idealized ice shelf geometry (illustrated in Fig.

1) ~~,~~ where the upper portion has calved away, emulating the “footloose” mechanism proposed by Wagner et al. (2014) where ~~a~~ an aerial portion of the calving front first detaches.

2 Problem Description

2.1 Glacier Stress Balance

Denoting the velocity field in two dimensions by $\boldsymbol{u}(x,z,t) = (u_x(x,z,t), u_z(x,z,t))$ and pressure by $P(x,z,t)$, conservation of linear-momentum can be written in the form:

$$\nabla \cdot \boldsymbol{\sigma} + \rho_i \boldsymbol{g} = \rho_i \frac{D\boldsymbol{u}}{Dt} \text{ ., } \tag{1}$$

where D/Dt denotes the material derivative. The Cauchy stress is defined in terms of strain rate, effective viscosity, pressure, and the identity matrix ~~I:~~
$\boldsymbol{I}$:

$$\boldsymbol{\sigma} = 2\eta\boldsymbol{\epsilon} - \underline{PI}P\boldsymbol{I}, \tag{2}$$

with strain rate tensor ~~ϵ~~ $\boldsymbol{\epsilon}$ given by:

$$\boldsymbol{\epsilon}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial r_j} + \frac{\partial u_j}{\partial r_i}\right). \tag{3}$$

Here ρ_i is the density of ice, $\boldsymbol{g}$ is the acceleration due to gravity, and η is the effective viscosity of ice:

$$\eta = \frac{B}{2}\epsilon_e^{\frac{1}{n}-1}. \tag{4}$$

The effective viscosity is a function of the effective strain rate ~~$\epsilon_e = \sqrt{\epsilon_{ij}\epsilon_{ij}}/2$~~ $\epsilon_e = \sqrt{\boldsymbol{\epsilon}_{ij}\boldsymbol{\epsilon}_{ij}/2}$, a temperature dependent constant B, and the flow-law exponent $n = 3$; the acceleration term on the right hand side of Eq. (1) denotes the material derivative.

In the Stokes ~~limit~~approximation, we drop the acceleration term from Eq. (1), ~~an approximation~~ which is justified for most glaciological applications ~~(Greve and Blatter, 2009)~~(e.g., Greve and Blatter, 2009). However, as we shall show, this assumption is problematic for applications where the ice departs from hydrostatic equilibrium.

2.2 Boundary Conditions

To illustrate an example where the Stokes flow problem ~~becomes ill-posed~~departs from hydrostatic equilibrium, we consider a two-dimensional floating ice shelf (Fig. 1). We specify the normal component of the velocity $\boldsymbol{u} \cdot \hat{n} = u_1$, where u_1 is a constant along the inflow portion of the domain (Γ_1) and $\hat{n}$ is the normal vector along Γ_1. At the ice-atmosphere boundary (Γ_2) we assume the surface is traction free. At the boundary between ice and ocean (Γ_3) the shear traction along the ice-interface vanishes and continuity of normal traction along the ice-ocean interface can be written as $\sigma_{nn}(x,t) = -\rho_w g b(x,t)$ where $b(x,t)$ is the position of the ice-ocean interface.

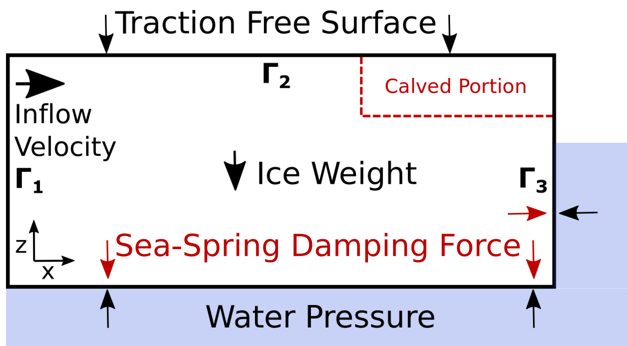


Figure 1. A diagram showing the boundary conditions of an idealized floating ice shelf. The ice-ocean interface is subject to two normal stresses - the depth dependent water pressure and the numerical damping force for stabilization to hydrostatic equilibrium. The dashed red line illustrates an iceberg that breaks off from the top of the calving front (exaggerated), reducing the freeboard and instantaneously perturbing the ice shelf from hydrostatic equilibrium.

Problems arise with this form if the glacier is not *exactly* in hydrostatic equilibrium because buoyancy forces along the ice-ocean interface cannot be balanced by internal stresses. In this case, there is no unique solution and vertical velocities are singular. In reality, of course, the ice will quickly re-adjust to hydrostatic equilibrium ~~through rapid~~ as a consequence of buoyant uplift through the (nearly) inviscid ocean.

We can more ~~accurately~~ appropriately specify the boundary condition for Stokes flow at the ice-ocean interface by writing it in the form:

$$\sigma_{nn}(x,t) = -\rho_w g\left[b(x,t) + \Delta z(x,t)\right] \quad \text{on} \quad \Gamma_3 \tag{5}$$

where $\Delta z(x,t)$ is an a *priori* unknown ~~uplift that~~ isostatic adjustment that could potentially include a rigid body translation and, for a freely floating iceberg, rigid body rotation. Crucially, the rigid body motion must be determined as part of the solution to enable the ~~full~~ local and global force balance to close.

The additional ~~uplift~~ isostatic adjustment term Δz has a simple physical explanation: if normal stress was exactly hydrostatic, $\sigma_{nn} = \rho_i g H$ where H is the ice shelf thickness. Equation (5) can then be solved for Δz to determine the position of the bottom interface needed for the forces to balance~~. The Stokes limit~~, which is exactly what is done in the shallow shelf approximation. The full Stokes approximation is more complex as internal stresses also contribute to the normal stress at the ice-ocean interface, but the location of the ice-ocean interface needs to be solved for as part of solution to the problem, which we examine next.

2.3 Numerical Stabilization of Buoyant Uplift

Different numerical methods ~~apply~~ use different techniques to ~~solve for~~ estimate Δz in Eq. (5). For example, in Elmer/Ice, a popular package for modeling Stokes glacier flow, Durand et al. (2009) proposed an ingenious solution in which Δz is

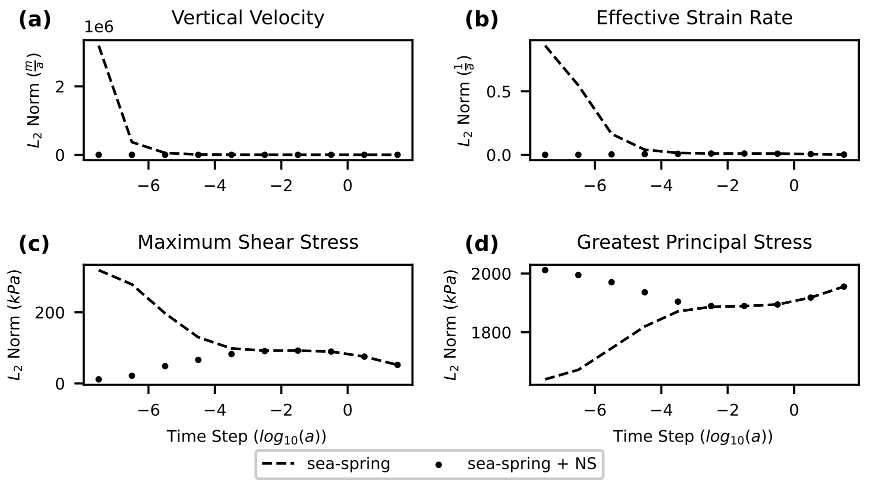


Figure 2. L_2 norm of **(a)** vertical velocity, **(b)** effective strain rate, ~~**(b)** viscosity,~~ **(c)** ~~effective~~ maximum shear stress, and **(d)** greatest ~~principle~~ principal stress immediately after the emulated calving event. Norm is calculated based on cell-averaged values. Solutions shown for sea-spring damping and sea-spring with Navier Stokes (NS) ~~. For small~~ for time steps ~~, the sea-spring solution diverges~~ranging from 1 second to 30 years. The sea-spring with Navier Stokes term ~~is well posed~~ gives physical solutions for all time steps ~~and~~ but still shows variability with time step consistent with the evolution of the system.

estimated based on a Taylor series of vertical position of the ice-ocean interface:

$$\Delta z = u_z(x,t)\Delta t + O(\Delta t^2). \tag{6}$$

This Taylor series transforms the ~~uplift~~ isostatic adjustment into a time ~~step dependent~~ step-dependent Newtonian velocity damping:

$$\sigma_{nn}(x,t) = -\rho_w g\left[b(x,t) + u_z(x,t)\Delta t\right]. \tag{7}$$

~~However, the~~ Here, the velocity u_z can include rigid body translation. The coefficient of the damping force in this approximation is proportional to the time step size Δt and vanishes in the limit of small Δt. In this small time step limit, vertical velocities ~~are~~ can become singular. We refer to the damping method given in Eq. (7) as the "sea-spring" method based on the nomenclature used in Elmer/Ice documentation.

~~With this method, we can decompose the velocity $\boldsymbol{u}$ into a "viscous" component $\boldsymbol{u}_{visc}$ and a hydrostatic uplift component $\boldsymbol{\Delta z}_{uplift}$, which we write in the form:~~

$$\boldsymbol{u}(\Delta t) = \boldsymbol{u}_{visc} + \frac{\boldsymbol{\Delta z}_{uplift}}{\Delta t},$$

~~where $\boldsymbol{\Delta z}_{uplift}$ is~~ We illustrate the time step dependence using a floating ice shelf as an example. In this case, global force balance is not guaranteed, leading to a formally ill-posed problem. However, the ~~vector form of the same displacement written~~

~~in Eq. (5) . Due to the dependence~~ problem of singular vertical velocities persists even when grounded ice is included in the domain because the velocity (and strain rate) solution can become unphysically sensitive to the position of the ice-ocean ~~boundary stress on time step size Δt , the total velocity becomes time step dependent~~interface.

~~Inspecting Eq. (??) shows that in the limit of large Δt, the uplift term becomes small compared to the viscous velocity. Thus, the sea-spring damping method can provide a good approximation for the viscous velocity as long as $(\boldsymbol{\Delta z}_{uplift})/(\boldsymbol{u}_{visc}\Delta t) << 1$, which is true if the glacier is close to hydrostatic equilibrium ($\boldsymbol{\Delta z}_{uplift}$ small) or a sufficiently long time step is used.~~

3 Calving-Based Convergence Test

3.1 Test Design

For our test, we implement an idealized rectangular ice shelf of thickness 400 m and length 10 km. This ice shelf is initialized to be in exact hydrostatic equilibrium~~with~~. We set the inflow velocity for the upstream boundary condition ~~set~~ to 4 km a^{-1}. These thickness and velocity parameters are broadly consistent with observations for the last 10 km of Pine Island ice shelf (Rignot et al., 2017, 2011; Mouginot et al., 2012; Paden et al., 2010, updated 2018). The temperature dependent constant in Glen's flow law is chosen to be 1.4×10^8 Pa s$^{\frac{1}{3}}$, the value given by Cuffey and Paterson (2010) for $-10°$C.

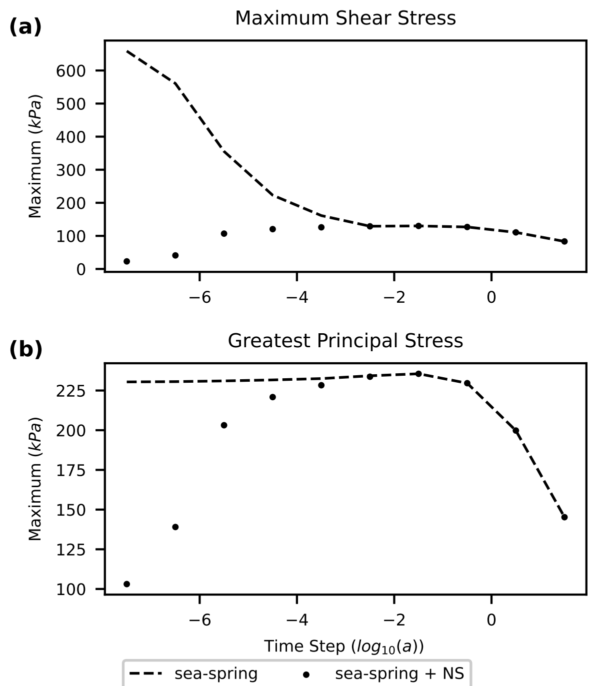
To emulate the occurrence of a calving event that would perturb the ice shelf from hydrostatic equilibrium, a rectangular section of length 50 m and thickness 20 m is removed from the upper calving front of the glacier (Fig. 1). This type of calving behavior has been proposed as the trigger of a larger calving mechanism related to buoyant stresses on the ice shelf (Wagner et al., 2014). The numerical effects we document are not unique to this style of calving and this mechanism is only meant to illustrate the numerical issues.

The problem is implemented in FEniCS (Alnæs et al., 2015), an open source finite element solver with a Python interface that has been previously used for Stokes glacier modeling (Ma et al., 2017; Ma and Bassis, 2019). The problem is solved using an Arbitrary Lagrangian-Eulerian formulation using mixed Taylor-Hood elements with quadratic elements for velocity and linear elements for pressure. The open source finite element mesh generator Gmsh is used to generate a unstructured mesh with uniform grid spacing of 10 m near the calved portion of the domain and grid spacing of 40 m elsewhere.

3.2 ~~Divergent~~ Time Step Dependent Behavior

Figure 2 shows the sensitivity of the ~~velocity field~~vertical velocity, effective strain rate, ~~effective~~ maximum shear stress, and greatest ~~principle~~ principal stress to time step size when using the sea-spring boundary condition from Eq. (7). ~~Furthermore, because of~~ The unphysically large velocities at small time steps are magnified by the coupling between effective strain rate and viscosity~~, the viscosity for the majority of the domain becomes unphysically small as the time step decreases. This positive feedback between effective strain rate and viscosity is especially problematic as higher strain rate causes lower viscosity, and vice versa, leading to unphysical results~~. As effective strain rates become larger with small time steps, viscosity decreases, causing even greater strain rates. This problem can be alleviated by using a viscoelastic rheology when examination of short

Maximum **(a)** effective shear stress and **(b)** greatest principle stress. Solutions shown for sea-spring damping and sea-spring with Navier Stokes (NS). Without corrections, maximum stress is overestimated in the Stokes model, which could lead to overestimation of glacier



retreat due to calving.

Figure 3. Maximum **(a)** maximum shear stress and **(b)** greatest principal stress immediately after the emulated calving event. Maximum is calculated based on cell-averaged values. Solutions shown for sea-spring damping and sea-spring with Navier Stokes (NS) for time steps ranging from 1 second to 30 years. Without corrections, maximum stress is overestimated in the Stokes model, which could lead to overestimation of glacier retreat due to calving.

time scale behavior is desired. However, even for a purely viscous model, short time steps may be necessary to satisfy numerical stability criteria during hydrostatic adjustment that momentarily forces the model outside of the Stokes range.

In addition to the divergence behavior of the L_2 norm, we also examine the maximum effective value of the maximum shear stress and greatest principle principal stress (Fig. 3). Maximum values may be a better predictor of the effect of time step dependence on the output of Stokes calving models. Because calving models often assume that calving is likely if a stress threshold is exceeded (Ma et al., 2017), outliers in stress are more important than a stress averaged over the entire domain.

4 Proposed Solution - Reintroduce Acceleration Term into Stress Balance

The velocity solution is ill-posed unphysical because the neglected acceleration term is not actually small relative to the other terms in Eq. (1):. This is because large velocities associated with hydrostatic adjustment rapidly change on time scales that are short compared to the internal deformation of the ice. We therefore It may be possible to separate a rigid body translation and rotation that satisfies global force and torque balance from the internal deformation, but this quickly becomes cumbersome and impractical when we include the potential for ice to break: global force and torque balance would have to be maintained on each intact portion of ice. Instead, we reintroduce the acceleration term directly to the Stokes equation and show that this regularizes the solution for small time steps. We use a simple first order backwards differentiation scheme in a Lagrangian reference framewhere :

$$\frac{D\boldsymbol{u}_i}{Dt} = \frac{\boldsymbol{u}_i - \boldsymbol{u}_{i-1}}{\Delta t}., \qquad (8)$$

This where $\boldsymbol{u}_i$ and $\boldsymbol{u}_{i-1}$ denote the velocity of Lagrangian fluid parcels at the current time step and the velocity at the previous time step, respectively. In the example considered here, the fixed horizontal velocity at the upstream boundary does not permit rigid body rotation and Equation 8 remains a valid approximation even in a Eulerian coordinate system. As an initial condition on velocity, we assume a uniform velocity field of 4 km a^{-1} (equal to the inflow velocity) in the horizontal direction and zero in the vertical direction. The choice of zero initial velocity in the vertical direction is consistent with the idea that the ice shelf has instantaneously been perturbed from hydrostatic equilibrium.

Restoring the inertial term effectively introduces a Newtonian damping term on the entire body of the glacier where the damping coefficient is $C = 1/\Delta t$. Computational difficulty is not impacted by reintroducing the acceleration term in this way because the term is linear with respect to velocity. However, unless a fully implicit scheme was implemented, the solution becomes inaccurate (and unstable) for long time steps. Therefore, we propose to use both damping terms so that the system of equations is numerically accurate for small all time step sizesand the velocity converges to the viscous limit for large time step sizes. Although this method rectifies the numerical inaccuracies present at short time steps with sea-spring damping, it does not address *physical* inaccuracies from using a rheology not suited for elastic effects. However, the numerical divergence exists independent of rheology and would need to be addressed even for a viscoelastic model. .

When we include both damping terms, vertical velocity, effective strain rate, effective principle maximum shear stress, and greatest principle stress are consistent principal stress maintain physical values for both small and large time steps (Fig. 2). At small time steps the acceleration term dominates , so and the sea-spring with Navier-Stokes Navier Stokes solution departs from the sea-spring solution. At In this limit, the (nearly) rigid body isostatic adjustment of the ice shelf dominates the solution. By contrast, for large time steps, the sea-spring damping dominates and the two solutions methods overlap. At intermediate time steps, both damping terms contribute as the solution transitions from the regime dominated by inertial effects to one where inertial effects are small. It is crucial to note that although the sea-spring with Navier-Stokes Navier Stokes solution retains time step dependence for small time steps, the time step dependence now results from the physical evolution of the system: the solution resolves the acceleration and deceleration of the glacier as it evolves towards bobs up-and-down in the ocean and approaches a steady-stateas opposed to being a consequence of an ill-posed problem.

Notably, Although the sea-spring solution shows larger maximum stresses a smaller L_2 norm of greatest principle stress than the sea-spring with Navier Stokes solution, this is due to large *negative* compressive stresses associated with being outside of hydrostatic equilibrium. If we instead examine the maximum of the stress fields, the sea-spring solution shows larger values for both maximum shear stress and greatest principle stress (Fig. 3). This is particularly evident for the ~~effective~~ maximum shear stress, which is overestimated by an order of magnitude at the shortest time step tested. In the footloose calving mechanism, when a portion of the upper calving front is removed, the front of the ice shelf becomes buoyant and produces increased shear stress upstream on the ice shelf (Wagner et al., 2014). This over prediction of stresses could cause a calving model to predict unphysical calving events due to numerical inaccuracies.

5 Conclusions

Our study shows that using a common numerical stabilization method of the ice-ocean boundary in Stokes glacier modeling there is an explicit time step dependence of the solution that ~~diverges~~ is unphysical for small time steps when the domain departs from hydrostatic equilibrium. For model applications where changes in the domain are only due to viscous flow, the time step dependence is not problematic as long as domains are (nearly) in hydrostatic equilibrium at the start of simulation. However, for applications where rapid changes to the model domain occur, such as when calving rules are implemented, sudden departure from hydrostatic equilibrium is not only possible, but expected. In these cases, time step dependence of the solution will appear. This can contaminate solutions of the stress after calving, potentially leading to a cascade of calving events and an overestimate of calving flux if numerical artifacts are not addressed. However, the time step dependence can be easily cured with little computational cost by reintroducing the acceleration term to the Stokes flow approximation. The acceleration term regularizes the solution for small time step sizes and ~~results in a physically consistent solution~~provides consistent solutions for all time steps.

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Competing interests. The authors declare that they have no conflicts of interest.

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