An analysis of instabilities and limit cycles in glacier-dammed reservoirs: response to reviewers

Christian Schoof

1 Referee # 1

This is an awkward paper to referee. Being Christian Schoof, there is not going to be anything technically wrong with it, but from my perspective it is not properly thought out.

Nor is it well presented. The very first thing to say is that the text is littered with typographical and grammatical errors, so many that I will not list them all here. But there are such errors on page 1, line 16; 1,24; 2,5; 2,6; 2,13; 2,15; 2,20 (twice); 2,29 (the whole second half); 2,35; and so on, and on, and on. Perhaps the best is saved for last, where Schoof's own 2012 paper is mis-referenced (it is part 1, not part 2). Incidentally, all the poor authors cited in the references are demoted to a single initial each.

I'm not sure my mere name warrants such confidence in the results, but I'll take this to mean I actually haven't made any mistakes. The apparent awkwardness is as difficult to answer at this point as it may have been for the referee to review the paper in the first place. In view of this, I will respond to the specific comments later. As far as the presentation is concerned— or rather, the number of typos and other superficial but annoying errors — I'm happy to concede the paper may have been submitted in too much of a hurry. My apologies for that. With regard to the author initials, I'll simply point out that I am using the Copernicus bibliography style file. As such the complaint is probably best addressed to the publisher.

The paper concerns a model (which is analysed in both a 'lumped' form and a spatially dependent one) for subglacial floods, or jokulhlaups. The paper is motivated heavily by previous work of Fowler and Ng, and seeks to modify this earlier work, by allowing for the case where there is no 'seal' of the subglacial (or ice-dammed) lake, which can then continually leak between floods, as is the case for Summit Lake, according to Fisher in 1973. Note: Fisher, not Fischer.

Fair point. I must have allowed my teutonic roots to influence my spelling.

The improvement consists of showing that with an extra term in the closure equa- tion of the classical Nye-Rothlisberger theory, the model will describe limit cycles even in the absence of a seal. This seems to me the principal achievement of this paper. The extra term invoked is an ingenious addition due to Schoof in 2010 which allows the description of both cavity drainage and channel drainage within the confines of a single model. It is worth offering some comments on this addition.

Indeed. I'd hope that beyond the qualitative statement that limit cycles are possible by adding a mechanism by which the drainage system remains 'open' in the refilling phase, an analysis of how flood magnitude and timing depends on forcing and geometry a valuable, too.

In its original form, the extra term appears as the first term on the right hand side of the closure equation

$$\dot{S} = u_b h_r + c_1 \Psi Q - c_2 N^n \left(\frac{Q}{c_3 \sqrt{\Psi}} \right)^{0.8}.$$

and it describes the opening of cavities by ice flow (velocity u_b) over bedrock bumps (height h_r). The steady state of this equation provides for both channels (N increases with Q) and linked cavities (N decreases as Q increases). There are several comments to make.

(i) We might suppose in reality that ub will itself depend on N as well as basal shear stress τ_b; might this not then ruin the conclusion? The answer is no, at least for sliding laws of power law type.

This is true; I have deliberately skirted this issue as the kind of glacier junctions at which dams are likely to occur often have awkward geometries — this is certainly true for Summit Lake, or the field site at the Kaskawulsh Glacier that has motivated my own interest in this subject (and before you ask: I do intend to publish data from said site, but inclusion in this paper would sure;y break the bounds of what is reasonable for paper length, even if I were to shorten the analysis). Those awkward geometries matter in the sense that treating ice flow as a function of local N might be stretching credulity. As the reviewer rightly points out, making u_b dependent on N doesn't ultimately break the mechanism being investigated, so long as u_b remains bounded below by something greater than zero. It just makes the flood cycle even less simple to describe. I'd be happy to add a brief discussion to the supplementary material if desired, but don't think this will add much to the main paper.

(ii) Second, Schoof's 2010 paper indicates a minimum value of N 2.6 MPa. This seems very high, and particularly seems unable to explain the very low values of N seen in the Siple Coast ice streams, for example. One might say these are sediment-floored, so that the concept of bed roughness is less clear to understand: does this mean one must abandon this theory in that context? The reason I enquire is that it seems to me that the understanding of sub-ice sheet floods is something this theory should aspire to.

This, I believe, is actually a reference to the fact that classical R-channel models with Nye (1953)-type closure rates consistently overpredict effective pressures, not just in the sense that they would require effective pressures larger than overburden, but in the sense that *measured* overburden pressures are usually significantly smaller. That is not an original observation of mine. The main reference to this that I'm aware of is Hooke, R.LeB. and Laumann, T. and Kohler, J., 1990, Subglacial water pressures and the shape of subglacial conduits. *J. Glaciol.* 36(122), 67–71. The point is that flatter channel shapes allow smaller effective pressures to balance predicted dissipation rates in the flow, and therefore to reconcile observation and theory. In the context of the model being used here, this issue is discussed at length in Schoof (2014) as cited in the present manuscript.

I agree that this is an appealing direction to develop outburst flood models in, and in particular, that the question of what the lateral aspect ratio of a channel actually is deserved further attention (ideally building on the vastly underappreciated D.Phil. thesis by Felix Ng.) Probably beyond the scope of this paper, though. As far as the concept of bed roughness becoming nebulous for deformable beds is concerned, I'm inclined to agree for relatively fine-grained beds with a narrower grain size distributions — as would apply for the formerly submarine bed areas of West Antarctica, for instance. For polydisperse grain size distributions, where there are larger cobbles and boulders mixed into the till, I'd argue that there are likely to be bed protrusions that can support cavities as in the canonical hard bed picture.

(iii) The boundary condition N = 0 is applied at the glacier snout. This is problematic because the closure equation then predicts S increases indefinitely. In the Schoof (2014) paper this is circumvented by saturating the opening term as $u_b h_r(1 - S/S_0)$, allowing for drowning of roughness at conduit size S0 ; this allows a steady state to be reached, but one in which $S > S0$, which makes no physical sense. In fact, the issue with the boundary condition is that the outlet flow must become open to the atmosphere at some unknown point upstream of the snout, where I think two boundary conditions should be prescribed for N and N_x , corresponding to continuity of water pressure and water flux. This may be important in view of figures 8 and 9, for example.

What is really at issue here I think is the way the cavity opening rate "goes away" for large S (note that this is only relevant to the spatially extended model, so I will restrict my discussion to the latter). In the model as posed, as in the earlier Hewitt et al (2012), Schoof (2010) and Schoof et al (2012, 2914) papers, this is done by writing the opening rate as $u_b(h_r - h)/l_r)$ for a continuum "sheet", or equivalently as $u_b h_r(1 - S/S_0)$ for an individual conduit. This does have the somewhat unintended consequence of leading to the opening rate becoming negative in an unbounded way when S exceeds the threshold S_0. A better way of dealing with the idea that bed roughness cannot indefinitely lead to a constant opening rate as conduit size grows (which is probably robust) might be to write the opening rate as $u_b h_r f(S/S_0)$ with $f(x) \to 1$ as $x \to 0$ and $f(x) \to 0$ as $x \to \infty$; something like $f(x) = (1 + \tanh(x))/2$ would do.

Having made the choice of cut-off function we have made here (where f goes to ∞ linearly as $x \to 0$ instead of vanishing), we can ask what difference this makes. In a model where cavity opening vanishes for large conduit sizes, the dominant balance near a glacier margin where $N \to 0$ would be between the melt rate $c_1 Q\Psi$ and closure rate $c_2 S N^n$. This would still leave the problem singular at the margin with $S \to 0$, but not in a pathological way (and the problem could further be regularized to maintain finite S by supposing that the glacier ends in a cliff so N is small but finite, or by supposing that the channel evolution equation is not cast in terms of $\partial S/\partial t$ but the material derivative $\partial S/\partial t + u_b \partial S/\partial x$.[1]

For that case, we can construct a near-margin form of the solution, and further ask how different the solution to other versions of the model is. In particular, for a pure channel model with a vanishing cavity opening rate, we get

$$S_t = c_1 Q_\Psi - c_2 S|N|^{n-1}N, \qquad Q = c_2 S^\alpha |\Psi|^{-1/2}\Psi, \qquad \Psi = \Psi_0 + N_x$$

and given a fixed discharge Q, a steady-state near-terminus solution can be constructed by noting that

$$\Psi = (Q/(c_3 S^\alpha))^{1/2}$$

so

$$c_1 c_3^{-2} Q^3 S^{-(2\alpha+1)} = c_2 N^n$$

and hence

$$N_x = -\Psi_0 + c_1^{-2\alpha/(2\alpha+1)} c_3^{-2/(2\alpha+1)} Q^{(1-4\alpha)/(2\alpha+1)} (c_2 N^n)^{2\alpha/(2\alpha+1)}$$

[1] This idea is due to Ian Hewitt, who may indeed have published it somewhere. While unappealing for cavities that are tied to bed roughness, the advection term must play a role near the snout, where melting happens not only because of subglacial water flow, but also from the surface, and ice flow must compensate for that.

which is clearly solvable form $x < L$ with a boundary condition $N(L) = 0$; the near field behaves as

$$
\begin{aligned}
N \sim & \Psi_0(L-x) - \\
& \frac{2\alpha+1}{2\alpha n + 2\alpha + 1} c_1^{-2\alpha/(2\alpha+1)} c_2^{2\alpha/(2\alpha+1)} c_3^{-2/(2\alpha+1)} Q^{(1-4\alpha)/(2\alpha+1)} \Psi_0^{2\alpha n/(2\alpha+1)} (L-x)^{(2\alpha n+2\alpha+1)/(2\alpha+1)} \\
S \sim & c_1^{1/(2\alpha+1)} c_2^{-1/(2\alpha+1)} c_3^{-2/(2\alpha+1)} Q^{3/(2\alpha+1)} \Psi_0^{-n/(2\alpha+1)} (L-x)^{-n(2\alpha+1)}
\end{aligned}
$$

Clearly, we can see that N remains well-behaved (and indeed positive, so the channel need not be partially open to the atmosphere!) while S blows up in a power-law fashion. We can go further and ask what the stability properties of the channel-only problem look like in the near field and construct a linearization. This is best done by changing the dependent variable S into something that remains bounded. An obvious choice is

$$Y = S^{1-\alpha}$$

The dominant balance when linearizing the problem above about the steady state ($Y = \bar{Y} + Y'$, $N = \bar{N} + N'$, $\Psi = \bar{\Psi} + N'_x$, $Q = \bar{Q} + Q'$, where barred quantities are steady state solutions and primed quantities are small perturbations) works out to be

$$
\begin{aligned}
Y'_t \sim & \frac{3}{2} c_1 c_3 \bar{\Psi}^{1/2} N'_x \\
Q' \sim & \frac{\bar{Q}}{2\bar{\Psi}} N'_x
\end{aligned}
$$

The germane question with using different model formulations that do not suppress the cavity opening term as above is whether they lead to the same solution away from a small region near the margin. As in, does the "regularization" of the model make any difference? In view of the question about numerical results tin figures 8 and 9, the question I will try to address is whether discrepancies between the channel-only model advocated above and the model used in the paper become more pronounced at large water throughputs in the model, which is the parameter regime that these calculations look at. For more moderate throughputs, the good agreement between lumped and spatially extended model suggests the issue of what happens near the margin (which does not feature in the lumped model) becomes less relevant.

The model used in the paper replaces the above by

$$S_t = c_1 Q_\Psi + u_b h_r (1 - S/S_0) - c_2 S|N|^{n-1} N, \qquad Q = c_2 S^\alpha |\Psi|^{-1/2} \Psi, \qquad \Psi = \Psi_0 + N_x.$$

In order to look at the difference from the channel-only model obtained by putting $u_b h_r = 0$, I will scale this by defining

$$[S] = \left(\frac{[Q]}{c_3 [\Psi]^{1/2}} \right)^{1/\alpha}, \qquad [t] = \frac{[S]}{c_1 [Q][\Psi]}, \qquad [N] = \left(\frac{c_1 [Q][\Psi]}{c_2 [S]} \right)^{1/n}, \qquad [x] = \frac{[N]}{[[\Psi]}$$

where $[Q]$ and $[\Psi] = \Psi_0$ are assumed to be given. Putting

$$S^* = \frac{S}{[S]}, \qquad N^* = \frac{N}{[N]}, \qquad \Psi^* = \frac{\Psi}{[\Psi]}, \qquad Q = \frac{Q}{[Q]}, \qquad t^* = \frac{t}{[t]}, \qquad x^* = \frac{x}{[x]},$$

and immediately dropping the star decorations, the model becomes

$$S_t = Q\Psi + \delta - \nu S - S|N|^{n-1} N, \qquad Q + S^\alpha |\Psi|^{-1/2} \Psi, \qquad \Psi = 1 + N_x$$

where

$$\delta = \frac{u_b h_r}{c_1[Q][\Psi]}, \qquad \nu = \delta[Q]^{1/\alpha} c_3^{-1/\alpha}[\Psi]^{-1/(2\alpha)} S_0^{-1}.$$

We can repeat the exercise of finding steady states. Assuming without loss of generality that the scaled flux $Q = 1$, we find $\Psi = S^{-2\alpha}$ and

$$S^{-(2\alpha+1)} + \delta/S - (nu + N^n).$$

Hence

$$\begin{aligned} S =& (nu + N^n - \delta/S)^{-1/(2\alpha+1)}, \\ N_x =& -1 + (\nu + N^n - \delta/S)^{-1/(2\alpha+1)} \end{aligned}$$

which the channel only model replaces by

$$\begin{aligned} S =& (nu + N^n)^{-1/(2\alpha+1)}, \\ N_x =& -1 + (\nu + N^n)^{-1/(2\alpha+1)} \end{aligned}$$

We want to know whether for larger $|L - x|$, the full and channel-only models will agree. This will be the case provided N agrees between the two models, and the latter will be the case if the correction δ/S remains small compared with $nu + N^n$ as well as having $\nu \ll 1$. This will be the case so long as $S \sim \nu^{-1/(2\alpha+1)}$ near $x = L$ is large enough, in other words, if $1 \gg \nu \gg \delta/nu^{-1/(2\alpha+1)}$ or $\nu \ll \delta^{2\alpha/(2\alpha+1)}$. The definitions of δ and ν above show that $\nu/\delta^{-2\alpha/(2\alpha+1)}$ increases with $[Q]$, all other parameters being constant, so we would in fact expect closer agreement between full and channel-only models for large $[Q]$.

We can go further and look at the linearization of the problem, again in terms of the variable Y used above (or rather, its obvious dimensionless counterpart); the dominant balances when adding the cavity opening term become

$$\begin{aligned} Y_t' \sim& \frac{3}{2}\bar{\Psi}^{1/2} N_x' + \delta \bar{Y}^{1/(\alpha-1)} Y' \\ Q' \sim& \frac{1}{2\bar{\Psi}} N_x' \end{aligned}$$

By similar construction to the above, if the steady state converges to that for the channel-only model as $|L - x|$ becomes large, so will the linearized solution for small δ; in other words, the additional term due to cavity opening will remain a small correction. This suggests that the stability resutls in the main paper remain robust for large water throughput rates.

Now we come to the main issue with this paper, which lies in its style. The paper does not know whether it is for glaciologists or applied mathematicians. The message is in fact fairly simple: here is a modification of Nye which allows limit cycles, even in a lumped version, and allows leakage between floods. But the material is drawn out by over-elaborate interpretations and explanations, and veers off into dynamical systems language which is neither helpful or informative. Starting on page 6, there is a rather long-winded stability analysis, which descends by page 8 to undergraduate mathematics. The only explanation can be that this is meant for glaciologists; but my view is that if they want to learn this material they should do so in textbooks, not *in a research paper. And in fact, all you need is figure 4.*

It goes on: we get undergraduate discussion of Hopf bifurcation, which by page 14 has slowed to the point of somnolence. And on. The section on asymptotic solutions on page 17 is mostly out of place here. What I actually think should happen is that the paper should be rewritten in two versions: a longer mathematical one which goes to a more mathematical journal (but then suitably prunes the more elementary stuff)and a shorter glaciological version which punches out the results: which are the model and some of the figures really.

Style may be where the referee and I won't agree. I am happy to shorten some of the material in the paper where appropriate, such as the linear stability analysis. The existing text undoubtedly can be optimized in that sense, but I don't think that's the issue. I understand the rationale for splitting work between "mathematical" papers and "glaciology" papers. This has been practised by a number of researchers in the past (Hutter, Morland, Fowler etc.) and even I have been known to try. However, in my own experience, what happens is that these mathematical papers, to the extent that they are taken up by anyone, get cited by glacioloists, not by applied mathematicians or fluid dynamicists outside of glaciology. The only exception are perhaps those dealing with numerical analysis of glaciological partial differential equations. A brief trawl through an indexing website like Web of Science should confirm that impression.

In short, there seems little point in these separate mathematical papers for an imaginary specialist audience. At the same time, I do not believe in simply saying "here are our mathematical results, but you wouldn't really understand so we won't explain any of the detail", which is the risk I see in writing a "glaciological version". What I do see in glaciology is an increasing number of researchers who have solid background in physics or similar disciplines. These researchers have the ability to understand mathematical material but may need a more didactic approach than the simple assumption that they have not only taken a course in dynamical systems theory, but actually remember its contents. This is the audience I'd like to reach here. Yes, doing so may mean a more pedestrian pace for the fully-fledged mathematician as a reader, but there are few enough of those around that I'm disinclined to worry (except about the referee, who I assume is an applied mathematician). I should add: I understand that a paper is not a textbook, but slightly more explanation to get a point across does not go amiss, and I think the manuscript as submitted is honest about what is ultimately textbook material and what is not (although Stogatz may admittedly be a more suitable textbook for the target audience than Wiggins).

I would add that a 'didactic approach' to presenting mathematical material in glaciology has been taken previously, even where that material arguably has limited novelty in a global (as opposed to discipline-specific) sense: to name but one example, a number of papers published in the Journal of Glaciology around 2011 (primarily by Bassis and Dukowicz et al) have elaborated on the fact that Stokes' equations are equivalent to a minimization problem — something that had been known to applied mathematics and fluid dynamics at least since the 1960s, but was apparently not widely known in glaciology. Whether the referee (who presumably hails form an applied mathematics background) would regard those papers as giving an undergraduate introduction to the calculus of variations I can't tell, but these particular papers clearly have had some impact (with 8 and 30 citations, respectively).

An analysis of instabilities and limit cycles in glacier-dammed reservoirs: response to reviewers

Christian Schoof

1 Referee # 2

Referee comment: *This paper investigates the mathematical properties of a conduit model (R-channel + one linked-cavity) when the upper boundary condition is a reservoir of fixed surface area and recharge rate. It looks at reservoir sizes corresponding a range from large glacier dammed lakes to moulins. It finds that for a given reservoir size that there are two stable regimes: one when the reservoir drains through the linked-cavity (i.e. recharge is low) and the other, when recharge is high, it drains through a moulin-like configuration. In-between the reservoir drainage is unstable and in fact periodic (for most situations), i.e. lake outburst floods. I think the paper nicely illustrates and investigates the range of behaviours to be expected from ice dammed reservoirs. Whilst this range of behaviours (leaking lake, out-bursting lake, and moulin-like) has been known, it has not yet been described quantitatively; certainly not with this mathematical rigour. Thus this paper is a significant step forward. However, the paper is a bit on the technical side for a glaciological paper. This is nicely illustrated by the brutal subsection of the Discussion (Sec 5.2), where the un-expecting reader – suffering from formula-overload already – is again presented with a wall of maths. And all this "discussion" does not actually lead to any further notable points of Discussion. I recommend to publish this MS after Sec 5.2 has been banned to the supplement (or another publication) and the comments detailed below are addressed.*

I would like to retain some version of the material in sec 5.2 here as opposed to offloading it to another paper, since such a paper would be rather stunted. The purpose of 5.2 was two-fold: first, the stability analysis leans heavily on that already presented in section 3.1, so keeping it in the same paper (overall, be that in the main text or supplementary material) makes sense to me. Secondly, the earlier paper by Schoof et al (2014) provides the main motivation for the present paper, and there we discuss not only outburst floods from localized reservoirs but also the effect of spatially extended storage, so it is natural to tie together the analysis of the mechanisms invovled in causing unsteady flow.

To address the referee's concerns while still touching on the effects of distributed drainage, I have now included the original section 5.2 in the supplementary material, and included a shortened version in the main paper. That shortened version simply sets up the problem with distributed drainage and the stability analysis, which allows me to clarify how the stability analysis is changed from the original version in section 2, and I simply state the main result, namely that an unstable wave can be formed even if the conduit is channel-like. Hopefully this is a little less formula-overload.

Referee comment: *The different triggering mechanism should probably be discussed a bit further. Of interest is in particular that many lakes drain with different triggering mechanisms from outburst to outburst (e.g. Grimsvötn 1996 vs other years, Gorner Lake (Huss et al., 2007)). In the case of Gorner Lake, no observations can predict the triggering mechanism.*

I have updated the introduction to reflect more of the observational literature on outburst floods, particularly covering Grímsvötn, Hidden Creek Lake, Gornersee, and an unnamed lake at the Kaskawulsh Glacier:

One possibility for flood initiation is that the lake simply fills to the level at which the ice dam starts to float, and a sheet flow emerges between ice and bed that subsequently channelizes (Flowers et al, 2004). This may occur irregularly in same lakes such as Grímsvötn due to exceptionally large inflow rates to the lake, for instance during volcanic eruptions, or as part of a repeating flood cycle in others (Bigelow et al, 2020). Some lakes are however known to initiate outburst floods before they reach that flotation level.

[. . .]

Grímsvötn typically starts its flood when water levels are below flotation (Björnsson, 1992), without successive floods growing in amplitude, except when the flood results from a large increase in water input due to volcanic activity (Gudmundsson et al, 1997). To explain this behaviour, Fowler (1999) considers the effect of a water supply along the length of the channel on its evolution. Such a water supply can maintain a minimum channel size even between outburst floods, with flow of water being directed partly into the lake and partly down-glacier to the margin, provided the glacier has a geometrical 'seal'. As the lake fills, the flow divide inside the channel migrates towards the lake, and the flood begins when the divide reaches the edge of the lake.

While this mechanism successfully explains how limit cycles (stable, periodic oscillations in lake level) can emerge in the model, it also predicts that no water can leave the lake between floods. Tracer experiments conducted at Salmon Glacier in Canada (Fisher, 1973) demonstrate that lakes can leak continuously throughout their recharge phase. Summit Lake, dammed by the Salmon Glacier, also has a history of flood initiation at lake levels below the flotation level (Post and Mayo, 1971). At Gornersee in Switzerland, observations of water lake levels and inferences of lake water balance based on meteorological measurements over two consecutive years also suggest leakage and flood initiation below flotation level during one year, and flood initiation at or above flotation level during the previous year, potentially in the absence of any pre-flood leakage (Huss et al, 2007). Observations at Hidden Creek Lake (Anderson et al 2003) by contrast suggest no leakage prior to flood initiation, though that conclusion is again based on water balance estimates rather than direct tracer experiments.

Referee comment: *The paper by Kessler and Anderson (2004) should be discussed further, both in the Introduction and Discussion as it uses also a conduit model (linked-cavity + R-channel) and applies it to a lake drainage (their section 4.2). For instance, they also see the pre-drainage leakage.*

I have discussed this in the introduction now, stating the following:

Note that many elements of the model studied here were included in the earlier study of outburst flooding by Kessler and Anderson (2004), who were able to replicate leakage from a glacier-dammed reservoir before the onset of the outburst flood, but did not attempt to reproduce recurring floods. It is unclear their model would have been able to do so since, in common with the model due to Fowler (1999), the main trunk of their drainage system was set up as a pure R-channel.

I'm quite hesitant to discuss the paper further, because I am rather uncertain whether I would be able to reproduce their model without either the code (which, in 2004, would not have been archived as supplementary material or on a code repository) or an unambiguous mathematical statement of the problem that the authors are discretizing. There are a number of unorthodox elements to the model description in Kessler and Anderson that I am somewhat loath to go into here, as I'm not addressing those authors directly, but suffice it to say that the following:

1. I *think* but cannot be sure that their network topology is tree-like, with cavity-or-channel network edges only present on the branches of the tree but not the main stem, so all water ultimately has to go through the R-channel-like main trunk, which ought therefore to behave like the R-channel in Fowler (1999). In that case, it is however unclear if there is supposed to be an analogous, uninterrupted background water supply that will keep the channel open to a minimum size and cause flow divide migration as in Fowler (1999). Knowing that would be key to understanding the initiation mechanism for *repeated* floods instead of a single flood, the latter being continent on the choice of initial conditions (which I think aren't fully specified in the paper, at least I couldn't tell). I expect the authors didn't intend to run their model for multiple melt seasons, so there may not be a unique answer to the point I'm making here. That makes discussion of their results in the context of what my manuscript tries to talk about — repeated floods — a bit difficult beyond what the current introduction states.

2. I also think but not entirely sure that the unorthodox description of conduit cross-section evolution in their equations (3), (4) and (6) is meant to say that the *actual* rate of change of S, i.e. $\mathrm{d}S/\mathrm{d}t$, is a potentially selective sum of the right-hand sides in (3), (4) and (5), with the main trunk missing out the right-hand side of (3); as stated, it looks like $\mathrm{d}S/\mathrm{d}t$ is somehow strangely multiply-defined. The text in the present manuscript assumes this interpretation, which makes their conduits the same in principle as mine.

3. The description of updating h according to (1a) and (1b) in their model was also not entirely clear to me; it looks to me like (1a) and (1b) together ought to correspond to a discretized parabolic problem for h (there is a derivative of Q_{cc} with respect to a downstream variable x in (1b), and I think Q_{cc} is defined in terms of a gradient of h in (5), although the notion of a derivative is discarded there in favour of a difference, suggesting a discretization in x should equally be applied in (1b). The text however then goes on to state that h is updated sequentially from the node that intiially has the highest h to lowest h (their paragraph [11]). This would be appropriate for a hyperbolic rather than parabolic problem, so again I don't feel confident I could reproduce the model

I'd rather not get into discussing these points in detail in the present manuscript.

Referee comment: *The model used has a single cavity, but it could also be used with many cavities in parallel. The supp. of Schoof (2010) does this. Mention briefly what the impact would be, I suspect it would only be quantitative.*

Things get a touch more complicated here: multiple identical conduits in parallel in a lumped model as in the supplementary material to Schoof (2010) perhaps unexpectedly predict that the stable limit cycle involves all conduits behaving identically. The channelizing instability that causes one conduit to 'win' when a drainage system is forced not with constant net throughput of water does not work fast enough when the system is instead forced with a reservoir at the upstream end. It seems like all conduits return to basically the same size during the refilling phase, and, being buffered by the reservoir, don't actually compete effectively for flux (which conduits do under fixed net flux conditions as in Schoof (2010)).

A linear stability analysis by itself doesn't reveal that insight. More to the point, however, it seems like allowing the conduits to have different physical parameters — for instance, different bed roughness heights or tortuosities — can fix that problem (as in, cause a single channel-like conduit to drain the lake). I intend to investigate that more deeply as part of ongoing work into surge mechanisms, but feel it would not add into insight here, and if anything, confuse the reader because it is a slightly subtle point that ultimately doesn't make a big difference, unless you suppose all channels are indeed identical. (Rather strangely, for a 2D network, a single channel seems to drain the reservoir even if all the channels are identical, and I haven't fully figured out why yet).

Referee comment: *The fact that moulins are "small reservoirs" is only mentioned really late in the MS. Could/should this be mentioned earlier?*

I've expanded on this theme in the introduction, in the following paragraph:

The original motivation for the work in Schoof et al (2014) was to demonstrate that behaviour akin to outburst floods can occur in systems with limited or even distributed water storage, manifesting itself in the form of unforced water pressure oscillations. Such limited storage could in principle result from moulins or any other vertical shaft that can fill progressively with water as water pressure rises.such as a basal crevasse. Viewed from that alternative different perspective, the present paper analyzes a drainage model that has become widely adopted in the study of subglacial hydrology (Werder et al, 2013). We identify instabilities that spontaneously lead to self-sustained oscillations, describing the mechanisms behind them and delineating the regions in parameter space where they occur. This is likely to be useful in diagnosing the behaviour of such models, regardless of their specific application to large outburst floods emanating from easily identifiable glacier 'lakes'.

Referee comment: *It is not clear to me why Appendix A is there but most other extra calculations are in the supplement, in particular as the more detailed calculation of Appendix A is also in the supplement.*

Appendix A is there to make the paper a little more self-contained. I felt (and this perhaps aligns with referee # 1?) that the other material in the supplementary material is really designed to fill in detail that a reader familiar with basic theory of dynamical systems could reconstruct, should they really want to try (in the terms of the other review, this is material that I'm including to help out the more "glaciological" reader, which a "mathematical" reader would probably not need). Appendix A is a brief summary of the material that goes beyond that level — it provides a brief sketch of results in the main paper that I expect would not be totally straightforward to reconstruct for a "mathematical" reader without undue effort.

Referee comment: *Please run a spell checker over the MS!*

Yes. This seems to be a common complaint, my apologies!

P1, L7: delete "a"

sure

Referee comment *P1, L16: mention that a lake drainage can also terminate when the lake is completely empty. This should be mentioned at a later stage as well, stating that this is not not relevant for this MS (the bed is flat).*

I have added the following to the end of the first paragraph of the introduction:

Alternatively, the flood can terminate because the lake has run dry. Then conduit then becomes partially air-filled and closes with minimal flow going through it.

I have also added further discussion of the bounds on effective pressure implied by flotation and the lake running dry in the fourth paragraph of section 5.1, where I describe the obvious modification to the lumped model that would reflect these bounds.

Referee comment: *P1, L20 write: "magnitude and timing of the flood." As for hazard prevention knowing the timing is probably equally important to the magnitude.*

Done

Referee comment: *P2, L13: "directly directed" is awkward*

Done

Referee comment: *Eq 1a: state that the pressure dependence of the melting point is neglected*

Added the following to the introduction: *Equation (1a) also ignores the effect water pressure on the melting point, which affects the fraction of the dissipated heat available for melting of the conduit walls (Werder, 2014).*

Referee comment: *Eq 1c: I find this equation strange. For v_o (and v_c) no separate equation $v_o = v_o(S)$ is added either. Thus be consistent and just write $q(S, \Psi)$. Similarly in all later equations.*

The reason for doing this is in the form stated is that q is a variable, not a function. Specifically, there is a term $\partial q/\partial x$, which doesn't make sense if q is not a function not of x, meaning I'd have to write explicitly $\partial q(X, \Psi)/\partial x$, which gets ugoy — the inconsistency here is really in using the same letter for variable and function. I have changed this to make $\tilde{q}$ the function of x and kept q as the function of S and Ψ

Referee comment: *P2: would it make sense to somewhere define what a "conduit", a "channel" and a "cavity" is? For example on P7,L4 "conduit" is used signalling the use of the v_o term again. So an unexpected reader may trip there without a clear definition.*

Sure, I have now elaborated on this to the effect that a conduit is a generic drainage element, while channels and cavities are really limiting cases of a conduit that have different qualitative properties. At the start of section 2.1, I now state that:

[. . .] *Note that we use the term 'conduit' here to refer to a generic drainage element that can evolve dynamically to behave as an R-channel, in which dissipation and creep closure are the dominant mechanisms by which channel size changes, or as a cavity, in which opening due to sliding over bedrock and creep closure are dominant (Kessler and Anderson, 2004; Schoof, 2010).*

My mistake was probably being careless about the use of the word "channel" (e.g. p. 4, line 11) — I believe I have now consistently used the word "conduit" except where I really mean "channel" in the sense of an R-channel-type balance of opening and closing terms.

Referee comment: *P4, L5: write "background hydraulic potential"*

done

Referee comment: *P4, L24: "large lakes"* done

Referee comment: *P4, L28: This paragraph confuses me. Is this not obvious? If not, be explicit what is odd. If it is obvious, delete.*

Deleted

Referee comment: *P5, L1: "in general"*

this went out of the window with the paragraph in the previous query

Figure 0: A figure depicting the used geometry would be helpful.

I've added one; not sure how much it really clarifies

Referee comment: *P6, L28: "model"*

corrected

Referee Comment: *P7, L21: these "reservoirs" are, e.g., moulins. Why not state this here?*

There is of course the case of very large reservoirs too, which are probably not realistic but theoretically can be stabilized, so I don't want to make a one-to-one correspondence with moulins here. In any case, the relevant paragraph has disappeared as part of a shortening of the stability section advocated by the other referee.

Referee comment: *P8 (on this page line numbers are messed up), L5+2: q(S, N)*

indeed; the relevant section has been changed significantly but I have corrected the mistake.

Referee comment: *P8, first un-numbered Eq: this should be v_o not v_0 . Or is $v_0 = v_o(S)$? If so state.*

This is a simple typo, corrected.

Referee comment: *P9, L24-29: this describes again a moulin*

I am happy to mention moulins here, subject to the same caveat as above. The updated passage reads:

The second term in (11) is a stabilizing term that is inversely related to storage capacity. Its physical origin is the following: the sensitivity of conduit growth to perturbations in conduit size may be positive, potentially leading to unstable conduit growth. However, growth of the conduit also allows water to drain out of the system, which will increase the effective pressure N. As the effective pressure is increased, the hydraulic gradient $\Psi = \Psi_0 - N/L$ is reduced. This leads to less turbulent dissipation in the conduit as the conduit grows, and increased N further leads to faster creep closure of the conduit. Both of these will suppress further growth of the conduit. How strong this stabilizing effect is will depend on the storage capacity of the system, and on the length L of the flow path: a large storage capacity $\bar{V}_p$ or a long flow path L leads to a reduced stabilizing effect. In practice, we are likely to see stabilization.

textbfReferee comment: *Fig 1: replace "lake" with "reservoir"*

done

Referee comment: *Fig 1: parenthesis missing after "3.3"*

corrected

Referee comment: *P11, L11: again v_o*

corrected

Referee comment: *P12, L22–23: the "immediately" needs to be weakened here. According to fig. 3: lake is empty and starts filling at the point (70,10), then the lake is filling again but S still drops from 10m2 to 0.*

This is true, but the it takes a very short amount of *time* (rather than distance in phase space) for that gap in phase space to be covered. I have changes thus to "almost immediately"

Referee comment: *Fig.3: Split the second sentence at the "and"*

Good idea, done

Referee comment: *Fig.4: could the plotted Vp be added as horizontal lines to Fig. 6?*

No doubt they can.

Referee comment: *Fig.4: I don't understand what the line style "solid dashed" is supposed to be. I think the unstable periodic should be described as "dotted coloured"*

Indeed. I have no idea what I was thinking. "Dotted coloured" it is.

Referee comment: *Fig.4: "insets"*

corrected

Referee comment: *Fig.5: zoom to relevant q_{in} values*

Done

Referee comment: *Fig.6: it is not clear to me what is meant with "as well as in a small strip to the right of the right-hand branch of the red curve." nor is this intriguing strip ever mentioned in the text. Clarify. Maybe a zoomed inset?*

I have added the following clarifying text:

[. . .] *as well as in a small strip to the right of the right-hand branch of the red curve: this is the region where the stable limit cycles shown for instance in the inset of figure 5(b) coexist with stable steady state solutions, and exist for larger q_{in} thn the periodic orbit that reaches a minimum of $N = 0$ near the upper critical value of q_{in}.*

Because the red curve is the locus in parameter space of periodic solutions that just reach $N = 0$ (rather than the boundary of the region in which $N < 0$ is attained by a limit cycle, which would be harder to compute), a zoom won't show anything here.

Referee comment: *Eq 16: should be "$\sim$"*

corrected

Referee comment: *P19, L16: "the its", delete "the"*

corrected

Referee comment: *Fig 7: "(b), 10 km"*

corrected

Referee comment: *Fig 7: "plotted in black" needs to be more specific. "plotted as black line"*

corrected

Referee comment: *Fig. 8/9, P32, L5: In view of this model behaviour and the potentially unstable numerics, some words should be said about the employed numerical methods (spatial discretisation, time-stepper). Yes, the code is provided but this should be in the text.*

I have added the following to the end of the first paragraph of section 4:

We adapt that method for the transient nonlinear calculations in figures 9–11 by solving equations (B1)–(B2) of the same appendix in Schoof et al (2014) using a backward Euler step.

Referee comment: *Fig 9: What is* $S_{\mathcal{R}}$ *?*

It's an idiotic legacy notation error on my part; it should say S.

Referee comment: *P23, L8: I don't see: "amplitude slowly grows" in fig 10*

Ok, "slowly" may be an overstatement. Changed to … *first grows and then saturates*

Referee comment: *P25, L1: "reservoir" instead of "lake"*

thanks for spotting this, corrected

Referee comment: *P26: twice wrong reference to 2012 instead of 2014*

corrected

Referee comment: *P26, L6: "results", P26, L21: "a spatial"*

"results" has been corrected. "to have spatial structure" seems ok sans the article.

Referee comment: *P27, L11:* $\Re$ *needs to be defined*

This has gone as part of a rewrite of Sec 5.2

Referee comment: *P30, L5–9: remove if Sec 5.2 is removed*

see above, I've kept this since I merely shortened section 5.2

Referee comment: *Supplement: Excellent, that the code is published! Two things: (1) There should be a README in each zip file, stating at least which script needs to be run to produce which figures. (2) I would suggest to add a licence to each zip-file (preferably an approved open source licence, the BSD-licence is popular with Matlab files https://opensource.org/licenses/BSD-3-Clause). Then it is clear under which conditions the code can be used.*

I have added README files to both directories. Note that the code for the transient calculations is actually identical to that used in Rada and Schoof (2018) and the code availablility statement now refers to that paper.

An analysis of instabilities and limit cycles in glacier-dammed reservoirs

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Abstract. Glacier lake outburst floods are common glacial hazards around the world. How big such floods can become (either in terms of peak discharge or in terms of total volume released) depends on how they are initiated: what causes the runaway enlargement of a subglacial or other conduit to start the flood, and how big can the lake get before that point is reached? Here we investigate how the spontaneous channelization of a linked-cavity drainage system ~~controls~~ can control the onset of floods. In agreement with previous work, we show that floods only occur in a band of water throughput rates in which steady reservoir drainage is unstable, and identify stabilizing mechanisms that allow steady drainage of an ice-dammed reservoir. We also show how stable limit cycle solutions emerge from the instability, ~~a show how and why the stability properties of a drainage system with spatially spread-out water storage differ from those where storage is localized in a single reservoir or 'lake'~~and identify parameter regimes in which the resulting floods cause flotation of the ice dam, and are therefore likely to be initiated by flotation rather than the unstable enlargement of of a distributed drainage system.

1 Introduction

Glacier lake outburst floods or jökullhlaups are a glacial hazard in many parts of the world (e.g. Björnsson, 1988; Clague et al., 2012). In addition, they provide a window into subglacial drainage systems: most outburst floods involve the opening and closing of a subglacial conduit, driven by melting of its walls through heat dissipation in the turbulent flow of water, and by viscous creep closure of the ice. In the early stages of the outburst flood, wall melting dominates through a positive feedback in which conduit enlargement leads to faster water flow and therefore more dissipation of heat. Later in the ~~flow~~flood, creep closure ~~accelarates~~ accelerates as the lake level drops, eventually causing the conduit to close again, and terminating the flood. Alternatively, the flood can terminate because the lake has run dry. Then conduit then becomes partially air-filled and closes with minimal flow going through it.

While the mechanics of the main discharge phase of an outburst flood are relatively well-understood (Nye, 1976; Clarke, 1982, 2003; Fowler and Ng, 1996; Ng, 1998; Fowler, 1999; Kingslake and Ng, 2013; Kingslake, 2013), the mechanisms that initiate ~~that~~ the flood are still poorly understood, even though they dictate the level to which the lake is able to fill, and therefore the magnitude and timing of the flood. In other words, the challenge is to explain not how a single flood progresses once started, but how it starts, and more to the point, what makes repeated floods occur cyclically, as a self-sustaining oscillation in the drainage system (Fowler, 1999; Kingslake, 2015). The main discharge phase of the flood is generally only part of a larger cycle, which also comprise a slow recharge phase in which outflow from the lake is small or absent altogether, and ~~crucially,~~ the transition between the ~~two, where the flood is initiated~~recharge and discharge phases, where both inflow into and outflow out of the lake are of comparable magnitude.

One possibility for flood initiation is that the lake simply fills to the level at which the ice dam starts to float~~on lake waters~~, and a sheet flow emerges between ice and bed that subsequently channelizes (Flowers et al., 2004). This may occur irregularly in same lakes such as Grímsvötn due to exceptionally large inflow rates to the lake, for instance during volcanic eruptions, or as part of a repeating flood cycle in others (Bigelow et al., 2020). Some lakes are however known to initiate ~~their~~ outburst floods before they reach that flotation level~~, which raises the question of what then starts the flow~~.

Motivated by the subglacial lake at Grímsvötn in Icealand, Ng (1998) and Fowler (1999) consider how outburst floods are initiated in a drainage system that consists of a Röthlisberger (R-) channel (Röthlisberger, 1972)~~controls the drainage and refilling of a lake over multiple Flood cycles~~. In the simplest version of their model, where the channel is fed purely by the lake, they find that the amplitude of floods grows from cycle to cycle, and ~~negative pressures~~ that water pressures exceeding ice overburden are eventually reached, meaning that floods should start through the flotation of the ice dam after a few cycles. At the heart of this behaviour is the fact that the R-channel can shrink to progressively smaller sizes between successive floods, making it harder to re-initiate drainage~~as the lake fills~~.

Grímsvötn typically starts its flood when water levels are below flotation (Björnsson, 1988), without successive floods growing in amplitude, except when the flood results from a large increase in water input due to volcanic activity (Gudmundsson et al., 1997). To explain this behaviour, Fowler (1999) considers the effect of a water supply along the length of the ~~channel~~ R-channel on its evolution. Such a water supply can maintain a minimum channel size even between outburst floods, with flow of water being ~~directly~~ directed partly into the lake and partly down-glacier to the margin, provided the glacier has a geometrical 'seal'. As the lake ~~level fills, that intra-channel flow divide~~ fills, the flow divide inside the channel migrates towards the lake, and ~~flood now initiates~~ the flood begins when the divide reaches the edge of the lake.

While this mechanism successfully explains how limit cycles (stable, periodic oscillations in lake level) can emerge in the model, it also predicts that no water can leave the lake between floods. Tracer experiments conducted ~~elsewhere (?)~~ at Salmon Glacier in Canada (Fisher, 1973) demonstrate that lakes can leak continuously throughout their recharge phase. Summit Lake, dammed by the Salmon Glacier, also has a history of flood initiation at lake levels below the flotation level (Post and Mayo, 1971). At Gornersee in Switzerland, observations of water lake levels and inferences of lake water balance based on meteorological measurements over two consecutive years also suggest leakage and flood initiation below flotation level during one year, and flood initiation at or above flotation level during the previous year, potentially in the absence of any pre-flood leakage (Huss et al., 2007). Observations at Hidden Creek Lake (Anderson et al., 2003) by contrast suggest no leakage prior to flood initiation, though that conclusion is again based on water balance estimates rather than direct tracer experiments.

In this paper, we consider an alternative mechanism by which floods can initiate in a recurring fashion without the need for a sealed lake, but with continuous leakage throughout the ~~flood~~ flooding-and-refilling cycle. We show that a conduit that is able to switch spontaneously between the behaviour of typical of an R-channel and the behaviour of a linked cavity system (Kessler and Anderson, 2004; Schoof, 2010; Hewitt et al., 2012; Hewitt, 2013; Werder et al., 2013) can sustain limit cycles without leading to flotation of the glacier at flood ~~initation~~initiation, and without requiring the flood divide migration appealed to in ~~(Fowler, 1999)~~Fowler (1999). Note that many elements of the model studied here were included in the earlier study of outburst flooding by Kessler and Anderson (2004), who were able to replicate leakage from a glacier-dammed reservoir before the onset of the outburst flood, but did not attempt to reproduce recurring floods. It is unclear their model would have been able to do so since, in common with the model due to Fowler (1999), the main trunk of their drainage system was set up as a pure R-channel.

~~This behaviour~~ The behaviour studied in the present paper was first documented in a model by Schoof et al. (2014). In this paper, we approach the problem from a more theoretical perspective, focusing on the following: ~~First~~first, we delineate when a 'lake' or other storage reservoir can be drained steadily, and ~~when~~ identify conditions under which steady drainage becomes unstable~~and when~~ , and how that instability leads to a limit cycle. Next, we investigate how the amplitude and period of floods depends on reservoir size and inflow rate. As a corollary, we determine at which point (in parameter space) the model predicts that flotation does occur and a different flood initiation mechanism is likely to take over. ~~Lastly, motivated by an extension of classical jökullhlaups models to spatially spread-out water storage in Schoof et al. (2014), we investigate the mechanism by which behaviour resembling outburst floods can occur when storage is not localized, showing that~~ In closing, we also briefly comment on the difference in flood dynamics caused by distributing water storage along the flow path rather than having a single water reservoir (Schoof et al., 2014). We outline how the classical runaway melt ~~effect~~ mechanism for outburst floods first described by Nye (1976) can be replaced ~~by another mechanism that relies on the introduction of~~ as a mechanism for self-sustaining water pressure oscillations by a spatial phase shift between conduit size and water pressure along the flow path.

The original motivation for the work in Schoof et al. (2014) was to demonstrate that behaviour akin to outburst floods can occur in systems with limited or even distributed water storage, manifesting itself in the form of unforced water pressure oscillations. Such limited storage could in principle result from moulins or any other vertical shaft that can fill progressively with water as water pressure rises.such as a basal crevasse. Viewed from that alternative different perspective, the present paper analyzes a drainage model that has become widely adopted in the study of subglacial hydrology (Werder et al., 2013). We ~~identifying~~ identify instabilities that spontaneously lead to self-sustained oscillations, describing the mechanisms behind them and delineating the regions in parameter space where they occur. This is likely to be useful in diagnosing the behaviour of such models, regardless of their specific application to large outburst floods emanating from easily identifiable glacier 'lakes'.

~~To do so, we~~ We use a hierarchy of ~~models~~model versions, employing both, a spatially-extended, ~~one dimensional~~ one-dimensional drainage system model and a lumped, box-type model that provides additional physical insight, as well as being the appropriate limit of the spatially-extended model in the case of a long flow path. The paper is laid out as follows: in section 2, we develop the spatially extended and lumped models. In sections 3.1 and 3.2, we identify instabilities in the lumped model~~, and where they are suppressed. 'Instability' here refers to the system evolving away from a steady state if the latter is slightly perturbed. What that instability evolves into~~ . The evolution of that instability at finite amplitude is investigated in detail in sections 3.3 and 3.4, where we show that stable limit cycles emerge in the lumped model. In section 4, we show that the lumped model replicates the behaviour of the spatially extended behaviour well in the classical parameter regime that corresponds to the drainage of large glacier-dammed lakes. We also show that the extended model is unstable in more exotic parameter regimes, where the lumped model is not, and investigate the resulting dynamics. These results are summarized in section 5.1, and we ~~take the results for localized reservoirs as motivation to study stability of drainage~~ briefly touch on the dynamics of systems with distributed ~~storage in 5.2. Our results are summarized in section 6, and extensive additional information are provided in~~ water storage in section 5.2, relegating detail to the supplementary material~~, including the code used in computations~~.

2 Model

2.1 A continuum model for outburst floods

We use the one-dimensional continuum model for drainage through a single conduit in Schoof et al. (2014), building on similar models used elsewhere (e.g. Ng, 1998; Hewitt and Fowler, 2008; Schuler and Fischer, 2009; Schoof, 2010; Hewitt et al., 2012). Note that we use the term ‘conduit’ here to refer to a generic drainage element that can evolve dynamically to behave as an R-channel, in which dissipation and creep closure are the dominant mechanisms by which channel size changes, or as a cavity, in which opening due to sliding over bedrock and creep closure are dominant (Kessler and Anderson, 2004; Schoof, 2010).

Denoting conduit cross-section by $S(x,t)$, effective pressure by $N(x,t)$ and discharge by $q(x,t)$, where x is downstream distance and t is time, we put

$$\frac{\partial S}{\partial t} = c_1 \underline{q}\tilde{q}\Psi + v_o(S) - v_c(S,N), \tag{1a}$$

$$\underline{\frac{\partial S}{\partial t} + \frac{\partial q}{\partial x}}\frac{\partial \tilde{q}}{\partial x} = \underline{rc_1 q\Psi}0, \tag{1b}$$

$$\underline{q}\tilde{q} = q(S,\Psi), \tag{1c}$$

$$\Psi = \Psi_0 + \frac{\partial N}{\partial x}, \tag{1d}$$

on $0 < x < L$, subject to boundary conditions

$$-V_p(N)\frac{\partial N}{\partial t} = q_{in} - \underline{q}\tilde{q} \qquad \text{at } x = 0, \tag{1e}$$

$$N = 0 \qquad \text{at } x = L, \tag{1f}$$

where L is the length of the flow path, and we assume the closures

$$v_o(S) = u_b h_r (1 - S/S_0), \qquad v_c(S,N) = c_2 S|N|^{n-1}N,$$

$$q(S,\Psi) = c_3 S^\alpha |\Psi|^{-1/2}\Psi. \tag{1g}$$

Here c_1, c_2, c_3, n, α, u_b, h_r and S_0 are positive constants with $\alpha > 1$, while q_{in} is also prescribed as a function of time and $V_p > 0$ is a function of N. Ψ_0 is a geometrically determined background hydraulic gradient, given in terms of ice surface elevation $s(x)$ and bed elevation $b(x)$ through

$$\Psi_0 = -\rho_i g \frac{\partial s}{\partial x} - (\rho_w - \rho_i) g \frac{\partial b}{\partial x}. \tag{1h}$$

Here ρ_w and ρ_i are the densities of ice and water, respectively, and g is acceleration due to gravity. Note that effective pressure N is linked to water pressures p_w, which is the primary observable in the field, through

$$N = p_i - p_w \tag{1i}$$

where p_i is overburden (or more precisely, normal stress in the ice at the bed, averaged over a length scale much larger than that occupied by the ~~channel~~conduit). Typically (including in (1h)), p_i is assumed to be cryostatic, equal to $\rho_i g(s-b)$.

Physically, the model represents a conduit whose size evolves due to a combination of dissipation-driven wall melting at a rate $c_1 q \Psi$, opening due to ice sliding over bed roughness at rate $u_b h_r(1 - S/S_0)$, and creep closure at rate $c_2 S N^n$, with discharge in the conduit given by a Darcy-Weisbach or Manning friction law as $c_3 S^\alpha |\Psi|^{-1/2}\Psi$ through (1g). u_b is sliding velocity, h_r is bed roughness and S_0 is a cut-off cavity size at which bed roughness is drowned out (Schoof et al., 2012). S_0 is typically a regularizing parameter that prevents conduits from becoming excessively large in regions where effective pressures are low, such as near the glacier terminus, but has little effect on conduit sizes along most of the flow path; this corresponds to the limit of large S_0. Equation (1a) also ignores the effect water pressure on the melting point, which affects the fraction of the dissipated heat available for melting of the conduit walls (Werder, 2014). We have also neglected the effect of water storage of water in the conduits, and of dissipation-driven melting, in the mass balance relation (1b); the reason for this is that a simple scaling exercise shows that these terms are invariably small for a terrestrial glacial system (supplementary material, section 2.2).

Vanishing effective pressure at the end of the flow path $x = L$ simply reflects the assumption that overburden and water pressure vanish simultaneously (where we have neglected atmospheric pressure as a gauge throughout). A computationally preferable assumption to $N = 0$ may be to assume that there is an ice cliff at the glacier terminus, so that vanishing water pressure still corresponds to a finite N and the effect of dissipation-driven opening does not have to be offset by an unrealistic closing of the conduit through the sliding term, which becomes negative when $S > S_0$; our results are however not sensitive to this assumption.

The upstream boundary condition at $x = 0$ ~~instead~~ reflects water storage in a reservoir at the head of the conduit: (1e) represents conservation of mass in the reservoir, with inflow rate q_{in}, outflow at rate $q(0,t)$ through the conduit modelled by (1a)–(1d) and with water storage V in the reservoir a function of effective pressure at $x = 0$ such that ~~$\mathrm{d}V/\mathrm{d}N = -V_p$.~~ $\mathrm{d}V/\mathrm{d}N = -V_p$ (see e.g. Clarke, 2003). Note that we deliberately use 'reservoir' rather than 'lake' here since we will be interested not only in large ~~lake~~lakes, but also in more modest-sized reservoirs as in Schoof et al. (2014). We will treat V_p and q_{in} as constants in most of what follows, although realistic lake shapes may argue for particular forms of the function $V_p(N)$ (Clarke, 2003). Conservation of mass along the conduit (1b) by contrast assumes that water storage is negligible along the

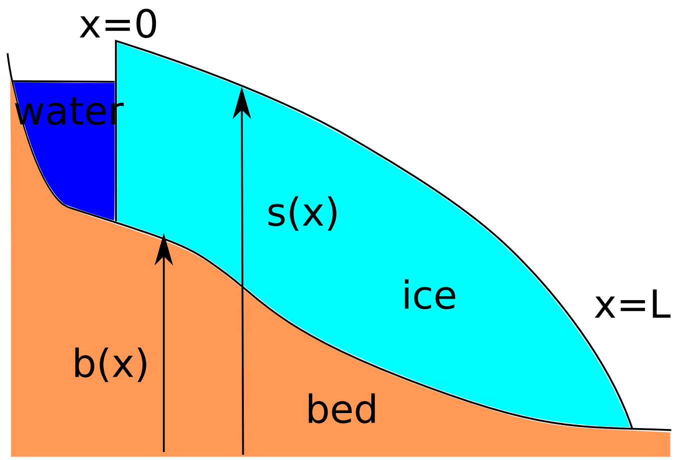

Figure 1. Geometry of the problem: the conduit runs along the ice-bed interface.

flow path. We show that the contribution of wall melting to mass flux is in general negligible in terrestrial drainage systems in section 2.2 of the supplementary material.

~~More generally, is justified as follows (Clarke, 2003). Water volume in the reservoir can be related to water depth h_w and hence to water pressure p_{w0} at the bottom of the reservoir as $\mathrm{d}V/\mathrm{d}h_w = A$, where A is the surface area that corresponds to the filling level h_w. With $p_{w0} = \rho_w g h_w$, $\mathrm{d}V/\mathrm{d}p_{w0} = A/(\rho_w g)$. In general, $A(h)$ is given by~~ For simplicity, we have neglected the possibility of a partially air-filled conduit, and the ~~shape of the reservoir. If we put $N = p_i - p_{w0}$ at the head of the conduit, then h_w and therefore A are functions of N at $x = 0$. Equation follows with $V_p = A(h_w)/(\rho_w g)$; constant V_p therefore corresponds to a lake with vertical sides, whose surface area is independent of filling level~~effects of overpressurization, where effective pressure becomes negative (Schoof et al., 2012; Hewitt et al., 2012). Overpressurization is particularly relevant to the initiation mechanism for subglacial floods where unstable enlargement of the conduit is not fast enough for the flood to occur before the lake reaches $N = 0$ and overfills. A partially air-filled conduit may form by contrast at the upstream end of the flood path if the lake fully drain at the end of the outburst floods, and could form more permanently at the downstream end as shown in Hewitt et al. (2012). We defer consideration of both processes to future work.

There is an important simplification we have made, irrespective of the particular choice of V_p: Our model effectively imposes an ice cliff at the conduit inlet $x = 0$, with the cliff height typically above the flotation thickness. This ensures that $N(0,t)$ can change over time without the upstream end of the conduit migrating. This contrasts with a glacier that partially floats on the lake, in which case the upstream end of the conduit is always at $N = 0$~~, but~~. In that case, the upstream conduit end migrates as the lake fills or empties, and ~~the location of that upstream end rather than effective pressure~~ its location is related to lake volume ~~. The filling level in the lake no longer dictates the effective pressure at the conduit inlet, but how much of the glacier has floated and therefore where that inlet has migrated to~~ (see section 2.4 of the supplementary material).

The assumption of an ice cliff, in addition, allows us to assume that $\Psi_0 > 0$ along the entire flow path, without a 'seal' at a finite distance from the reservoir at which Ψ_0 changes sign (Fowler, 1999). With a seal, there is a finite region of negative Ψ_0

between reservoir and seal location, with water flow out of the reservoir possible only when local effective pressure gradients $\partial N/\partial x$ are large enough to overcome the effects of the seal to make the total hydraulic gradient Ψ positive in (1d). In the absence of a seal and without an ice cliff (so $N = 0$ at the edge of the reservoir), normal stress at the glacier bed just outside of the reservoir would then be below the flotation pressure, and there would be nothing to dam the reservoir (see section 2.4 of the supplementary material).

The two situations (an ice cliff and a seal) can be reconciled in the sense that a seal very close to the edge of the reservoir corresponds to a short, steep ice surface slope rather than an actual cliff. This results in a large, negative Ψ_0 between reservoir and seal over a short distance, which can be balanced by an equally large $\partial N/\partial x$ over the same short distance, leading to a non-zero effective pressure at the seal itself, only a short distance from the lake.

Nevertheless, our simplifying assumption is relevant as the mechanism for initiating periodically recurring floods is fundamentally different from that in Fowler (1999): his model relies on a geometrical seal with negative Ψ_0 near the reservoir, changing sign at the seal location, and an englacial water supply to the conduit that dictates the gradient $\partial N/\partial x$ while the lake is filling. As described in the introduction, the onset of the outburst flood corresponds to the instant at which Ψ at the the conduit inlet $x = 0$ changes from negative to positive, and allows water to flow out of the reservoir. This leads to runaway enlargement of the conduit through dissipation-driven melting.

By contrast, our model assumes that the reservoir always experiences some amount of leakage, and requires no englacially-supplied conduit: the water supply to the drainage system in our model is purely through the inflow term q_{in} into the reservoir. Leakage out of the reservoir is facilitated by the linked-cavity behaviour of the conduit between floods, associated with the cavity opening term v_o that keeps the conduit open even when flow rates are small. This is absent from the models in Fowler (1999) and Ng (1998). As the lake fills, these cavities enlarge because the effective pressure is reduced, and the conduit then grows through the same melt-driven ~~channel~~ conduit enlargement as in Fowler (1999) and Ng (1998), but the initiation of that enlargement differs.

2.2 A lumped model

The model (1) can be simplified if we assume that Ψ_0 not only does not change sign along the flow path, but can be treated as a constant, and if we assume that the effective pressure gradient $\partial N/\partial x$ along the flow path can be treated as negligible (see also Fowler, 1999; Ng, 2000, and sections 2.2 and 2.3 of the supplementary material).

Under these assumptions, we can relate the flux q along the flow path (which is independent of position by (1b)) purely to the conduit size S and hydraulic gradient Ψ_0 at the head of the ~~channel~~conduit, leading to a system of ordinary rather than partial differential equations:

$$\dot{S} = c_1 \underline{q}\tilde{q}\Psi + v_o(S) - v_c(S, N) \tag{2a}$$

$$-V_p \dot{N} = q_{in} - \underline{q}\tilde{q} \tag{2b}$$

$$\underline{q}\tilde{q} = q(S, \Psi) \tag{2c}$$

where a dot signifies an ordinary derivative with respect to time, and we continue to assume the closure relations (1g); by the argument above, we then strictly speaking have to put $\Psi = \Psi_0$. We generalize this reduced model slightly to account qualitatively, if not quantitatively, for the effect of pressure gradients along the flow path by putting

$$\Psi = \Psi_0 - N/L, \tag{2d}$$

which corresponds to a crude divided difference approximation of the actual gradient $\partial N/\partial x$ by $[N(L,t) - N(0,t)]/L$

In what follows, we proceed first with an analysis of the simpler, ‘lumped’ model (2), to which we can bring to bear the theory of finite-dimensional dynamical systems ~~(Wiggins, 2003)~~(Strogatz, 1994; Wiggins, 2003), leading a number of semi-analytical results. Subsequently, we study the full, spatially-extended model (1), for which we are limited to comparatively expensive numerical methods that we can guide by our analysis of the simpler, lumped model.

3 An analysis of the lumped model

3.1 Nye’s jökulhlaup instability in the reduced model

Nye’s (1976) original theory of jökulhlaups centers on the idea that steady flow in ~~a channel~~ an R-channel is unstable if there is a water reservoir that keeps water pressure approximately constant. This instability results in the runway growth of a channel, eventually allowing the reservoir to drain in an outburst flood. The assumption of a constant effective pressure is of course a simplification of the more complete model (2), based on the notion that the drainage of the lake is generally too slow to lead to changes in effective pressure that could stabilize the channel against growth.

The basis of this instability is relatively easy to capture mathematically. To set the scene, we present a simplified version first before tackling an analysis of the complete ~~mdel~~ model (2). If we assume a classical ~~‘channel’~~ R-channel in the sense of Röthlisberger (1972) and Nye (1976) and therefore set the cavity opening term v_o to zero in (2a), and use the remaining relations in (1g), the conduit evolution equation (2a) becomes

$$\dot{S} = c_1 c_3 S^\alpha |\Psi|^{3/2} - c_2 S |N|^{n-1} N, \tag{3}$$

with a steady state conduit size $\bar{S}$ given by $c_2 \bar{S} |\bar{N}|^{n-1}\bar{N} = c_1 c_3 \bar{S}^\alpha |\Psi|^{3/2}$. At fixed effective pressure N, and hence at fixed Ψ, this steady state is unstable: an increase S' away from the steady state size $\bar{S}$ will lead to a further growth in he conduit, as we have $(\bar{S} + S')^\alpha \approx \bar{S}^\alpha + \alpha \bar{S}^{\alpha-1} S'$ and so

$$\begin{aligned}\dot{S}' &\approx c_1 c_3 \alpha \bar{S}^{\alpha-1} S' |\Psi|^{3/2} - c_2 S' |N|^{n-1} N \\ &= c_1 c_3 (\alpha - 1) \bar{S}^{\alpha-1} |\Psi|^{3/2} S',\end{aligned} \tag{4}$$

where, with $\alpha > 1$, the right-hand side is positive if S' is, signifying that S' will continue to grow in a positive feedback.

In more abstract terms, if Ψ and N are fixed, (2a) can be expected to lead to unstable growth of conduits away from an equilibrium state $\bar{S}$ if (see also pages 5–6 of the supplementary material to Schoof (2010))

$$c_1 \left.\frac{\partial q}{\partial S}\right|_{S=\bar{S}} \Psi + \left.\frac{\partial v_o}{\partial S}\right|_{S=\bar{S}} - \left.\frac{\partial v_c}{\partial S}\right|_{S=\bar{S}} > 0. \tag{5}$$

This is also precisely the condition that defines a conduit as being 'channel-like' in Schoof (2010), in the sense that two neighbouring conduits will compete for water with one eventually growing at the expense of the other if both are 'channel-like'. ~~A channel-like conduit is also distinguishable by the fact that the effective pressure required to balance opening processes (dissipation-driven wall melting and cavity opening due to flow over bed toughness) by creep closure increases with discharge q_{in} in the channel; the opposite is true for a 'cavity-like' conduit.~~

Key to the instability mechanism above was the notion that effective pressure N is kept constant as the conduit evolves, which is not actually the case: instead, the lake simply buffers changes in N from happening rapidly~~, since a change in N can only occur after an imbalance develops between inflow q_{in} and outflow q~~. Next, we investigate in more detail how water storage and the finite size L of the system control whether Nye's instability does occur in the reduced model of section 2.2 ~~. Our main result will be that this is not unconditionally the case, and that some reservoirs can drain steadily without outburst floods.~~

Note that v_o~~and~~ , v_c ~~satisfy~~ and q satisfy

$$\frac{\partial v_o}{\partial S} \leq 0, \qquad \frac{\partial v_c}{\partial S} > 0, \qquad \frac{\partial v_c}{\partial N} > 0, \qquad \frac{\partial q}{\partial S} > 0, \qquad \frac{\partial q}{\partial \Psi} > 0. \tag{6}$$

with $v_o(S) > 0$ bounded as $S \to 0$, $v_c(S,0) = 0$ and $v_c(0,N) = 0$, ~~while q satisfies~~

$$\frac{\partial q}{\partial S} > 0, \qquad \frac{\partial q}{\partial \Psi} > 0.$$

q also has the same sign as Ψ, and satisfies $q(0,\Psi) = q(S,0) = 0$. These are the minimal assumptions we make on these functions, allowing us to generalize from the specific forms in (1g) in analyzing Nye's instability.

The primary dependent variables in the model (2) are S and N. For constant water input q_{in}, the model admits a steady state $(\bar{S},\bar{N})$ given implicitly by

$$c_1 q(\bar{S},\bar{\Psi})\bar{\Psi} + v_o(\bar{S}) - v_c(\bar{S},\bar{N}) = 0 \tag{7a}$$

$$q(\bar{S},\bar{\Psi}) = q_{in} \tag{7b}$$

$$\bar{\Psi} = \Psi_0 - \bar{N}/L. \tag{7c}$$

~~If $q_{in} > 0$, it follows that $\bar{\Psi} > 0$, and from and the properties of v_c, we also have $\bar{N} > 0$. For future convenience, we also write $q(\bar{S}.\bar{N}) = \bar{q}$. Note that, for the specific choices in , a steady state exists for every positive q_{in}. Eliminating $\bar{S}$ and $\bar{\Psi}$, we find a problem for $\bar{N}$ alone:~~

$$c_1 q_{in}\left(\Psi_0 - \frac{\bar{N}}{L}\right) + v_0 - c_2\left(\frac{q_{in}}{c_3(\Psi_0 - \bar{N}/L)}\right)^{1/\alpha}\bar{N}^n = 0$$

~~The left-hand side is a monotonically decreasing function of $\bar{N}$ for $0 \leq \bar{N} < \Psi_0 L$, positive at $\bar{N} = 0$ and tending to $-\infty$ as $\bar{N} \to \Psi_0 L$, so the equation always admits a unique solution for $\bar{N}$ in this range, from which $\bar{S}$ can then be determined. The solution then has~~ where we assume positive lake inflow $q_{in} > 0$. It can be shown (see section 3 of the supplementary material) that there is a unique steady state solution with positive $\bar{N}$, $\bar{S}$ and $\bar{\Psi}$.

~~The point of the stability analysis is ultimately to establish conditions under whether this steady state is stableand can therefore persist over time: if so, no outburst floods need to result from the presence of the reservoir. Linearizing about~~ To establish whether the solution is stable, we can linearize around the steady state as

$$N = \bar{N} + N' \exp(\lambda t), \qquad S = \bar{S} + S' \exp(\lambda t).$$

~~and putting $\bar{V}_p = V_p(\bar{N})$ gives the following leading-order form of :~~

$$\dot{S}' = c_1 q_S \bar{\Psi} S' - (q_\Psi \bar{\Psi} + \bar{q}) L^{-1} N'$$

$$+ v_{o,S} S' - v_{c,S} S' - v_{c,N} N'$$

$$-\bar{V}_p \dot{N}' = -q_S S' + q_\Psi L^{-1} N'$$

This yields the following eigenvalue problem

$$\begin{pmatrix} c_1 q_S \bar{\Psi} + v_{o,S} - v_{c,S} - \lambda & -c_1 (q_\Psi \bar{\Psi} + \bar{q}) L^{-1} - v_{c,N} \\ \bar{V}_p^{-1} q_S & -\bar{V}_p^{-1} q_\Psi L^{-1} - \lambda \end{pmatrix} \begin{pmatrix} S' \\ N' \end{pmatrix} = 0, \tag{8}$$

where

$$q_S = \left.\frac{\partial q}{\partial S}\right|_{S=\bar{S},\Psi=\bar{\Psi}}, \quad q_\Psi = \left.\frac{\partial q}{\partial \Psi}\right|_{S=\bar{S},\Psi=\bar{\Psi}}, \quad v_{o,S} = \left.\frac{\mathrm{d} v_o}{\mathrm{d} S}\right|_{S=\bar{S}},$$

$$v_{c,S} = \left.\frac{\partial v_c}{\partial S}\right|_{S=\bar{S},N=\bar{N}}, \quad v_{c,N} = \left.\frac{\partial v_c}{\partial N}\right|_{S=\bar{S},N=\bar{N}}.$$

~~As before, we want to know whether S' grows over time. Looking for solutions of the form $S' = S_0' \exp(\lambda t)$, $N' = N_0' \exp(\lambda t)$, we get the eigenvalue problem~~

$$v_{c,S} = 0 \left.\frac{\partial v_c}{\partial S}\right|_{S=\bar{S},N=\bar{N}}, \quad v_{c,N} = \left.\frac{\partial v_c}{\partial N}\right|_{S=\bar{S},N=\bar{N}}, \quad \bar{V}_p = V_p(\bar{N}).$$

Setting the determinant of the matrix on the left~~to zero leads to a polynomial for λ,~~, we obtain a quadratic with solution

$$\lambda^2 - = \frac{1}{2}\left[a_1 \lambda + a_2 = 0 \pm \sqrt{a_1^2 - 4a_2}\right] \tag{9}$$

where ~~the coefficients take the form~~

$$a_1 = c_1 q_S \bar{\Psi} + v_{o,S} - v_{c,S} - \bar{V}_p^{-1} q_\Psi L^{-1} \tag{10a}$$

$$a_2 = \bar{V}_p^{-1} q_S c_1 (q_\Psi +) L^{-1} + v_{c,N} - {}_p^{-1} q_\Psi L^{-1} c_1 q_S + v_{o,S} - v_{c,S} = {}_p^{-1} \left[c_1 q_S \bar{q} L^{-1} + q_S v_{c,N} + q_\Psi L^{-1}(v_{c,S} - v_{o,S})\right] \tag{10b}$$

~~But, from~~ From our assumptions on the various functions involved, we see that $a_2 > 0$ (recall that $v_{o,S} \leq 0$ from (6)), while a_1 can be either sign.

~~The characteristic quadratic has solutions~~

$$\lambda = \frac{1}{2}\left[a_1 \pm \sqrt{a_1^2 - 4a_2}\right]$$

Since $a_2 > 0$, we have $a_1^2 - 4a_2 < a_1^2$ and two possible types of solution: either $a_1^2 - 4a_2 > 0$ and we have two real roots, both of which have the same sign as a_1. Alternatively, we have $a_1^2 - 4a_2 < 0$ and a complex conjugate pair of roots, both of which have real part a_1. In either case, we see that the system is linearly unstable if and only if $a_1 > 0$, or

$$\left(c_1 q_S \bar{\Psi} + v_{o,S} - v_{c,S}\right) - \bar{V}_p^{-1} q_\Psi L^{-1} > 0. \tag{11}$$

We have deliberately written the left-hand side of (11) as the difference of two terms, a potentially destabilizing term $c_1 q_S \bar{\Psi} + v_{o,S} - v_{c,S}$ and a stabilizing term $-\bar{V}_p^{-1} q_\Psi L^{-1}$. The first term can be recognized as the growth rate sensitivity we previously identified as being at the heart of Nye's instability in (5), as well as an indicator of whether the steady-state conduit is 'channel-like' in the terminology of Schoof (2010).

The second term in (11) is a stabilizing term that is inversely related to storage capacity. Its physical origin is the following: the sensitivity of ~~channel~~ conduit growth to perturbations in conduit size may be positive, potentially leading to unstable conduit growth. However, growth of the conduit also allows water to drain out of the system, which will increase the effective pressure N. As the effective pressure is increased, the hydraulic gradient $\Psi = \Psi_0 - N/L$ is reduced. This leads to less turbulent dissipation in the conduit as the conduit grows, and increased N further leads to faster creep closure of the ~~channel~~conduit. Both of these will suppress further growth of the conduit.

How strong this stabilizing effect is will depend on the storage capacity of the system~~. If the storage capacity is large, then the growth of the channel will have a minimal impact on effective pressure (essentially, the amount of water that drains out due to the widened conduit is insufficient to affect water levels and therefore water pressure in the storage system sufficiently). The stabilizing effect of the second term is therefore small when~~ , and on the length L of the flow path: a large storage capacity $\bar{V}_p$ ~~is large. Similarly, if the system size~~ or a long flow path L ~~is large, then the role of effective pressure in controlling hydraulic gradients is small, and Ψ is always close to the background hydraulic gradient. The stabilizing effectis then again small , as drawing water out of the storage system does not significantly affect hydraulic gradients and turbulent dissipation~~leads to a reduced stabilizing effect. In practice, we are likely to see stabilization for small storage elements, such as moulins and individual crevasses.

~~In summary,~~

3.2 Stability boundaries

We have established that Nye's instability will occur if the conduit is channel-like and water storage in the system is sufficiently large, and if the system length L is big enough. Note that with the choices in (1g), the system is always unstable if $v_o = 0$ (so the conduit is always a channel) and $L = \infty$ (so effective pressure does not alter the hydraulic gradient). This is in agreement with Ng (1998).

Parameter	value
c_1	1.3455×10^{-9} J^{-1} m^3
c_2	3.44×10^{-24} Pa^{-3} s^{-1}
c_3	4.05×10^{-2} m$^{9/4}$ Pa$^{-1/2}$ s^{-1}
α	5/4
n	3
$u_b h_r$	3.12×10^{-8} m^2 s^{-1}
S_0	170 m^2 (spatially extended model)
S_0	∞ (lumped model)
Ψ_0	178 Pa m^{-1}

Table 1. Parameter values common to all calculations except those in figure 12, for which $\Psi_0 = 1630$ Pa m^{-1} (corresponding to a much steeper 17:100 slope), $u_b h_r = 1.05 \times 10^{-07}$ m^2 s^{-1} and $L = 5$ km.

3.3 ~~Stability boundaries~~

~~So far, we have only established abstract conditions under which steady drainage through the conduit is unstable and will therefore not persist, potentially leading to periodic outburst floods.~~ We can go further and determine explicitly the regions of parameter space in which this stability occurs. While we present our results here in dimensional form, note that it is possible to reduce the spatially extended and lumped models (1) and (2) to a four-dimensional parameter space by non-dimensionalizing them (sections 2.2 and 2.3 of the supplementary material). These parameters are dimensionless versions of the inflow rate q_{in}, storage capacity V_p, system length L and conduit cut-off size S_0. Recall that S_0 is intended to have minimal impact on conduit evolution away from the glacier margin, and we are therefore interested in the limit of large S_0. As a result, we restrict ourselves to the three-dimensional parameter space spanned by $(q_{in}, \bar{V}_p, L)$ and set $S_0 = \infty$ in the lumped model (2). The remaining fixed parameter values we have used are given in table 1. The low value of Ψ_0 stated corresponds approximately to a 1:50 surface slope.

~~It can be shown analytically that there is at most a finite range of values of q_{in} for which the instability occurs for given $\bar{V}_p$ and L (section 3.1 of the supplementary material): when q_{in} is too small, the conduit is cavity-like as opposed to channel-like, so is not satisfied. By contrast, when q_{in} is too large, the stabilizing term $\bar{V}_p^{-1} q_\Psi L^{-1}$ in dominates. The intermediate range of inflow values q_{in} that makes the system unstable is not guaranteed to exist: a sufficiently large reservoir size $\bar{V}_p$ and a long flow path length L are required to prevent the stabilizing term from dominating. The range of unstable q_{in} values also increases as $\bar{V}_p$ and L increase.~~

~~This is confirmed by direct numerical computations of the stability boundaries~~ Figure 2 shows stability boundaries in the $(q_{in}, \bar{V}_p)$-plane for different values of L. These are the locations in parameter space where the real part of the growth rate λ in (9) is zero (see section 3.2 of the supplementary material for details)~~: figure 2 shows stability boundaries in the $(q_{in}, \bar{V}_p)$-plane for different values of L~~; invariably, the eigenvalues λ then form purely imaginary conjugate pair, and we have a so-called

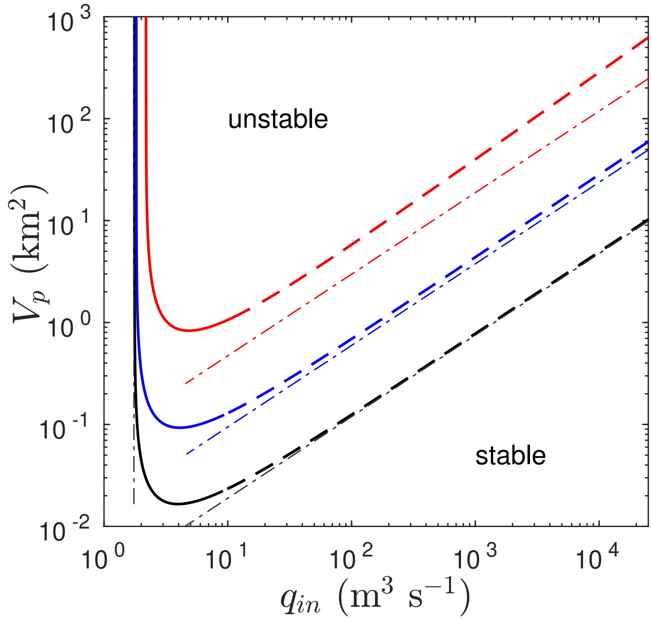

Figure 2. Stability boundaries for ~~lake~~ reservoirs systems with the parameter values in table 1 shown as solid/dashed curves. Note the logarithmic scales on the axes. The unstable region in parameter space is above the curves as shown. The curves correspond to different values of L: 10 km (red), 50 km (blue) and 250 km (black); the latter is not intended to be physically realistic but to reflect the limit of large L. The Hopf bifurcation at the stability boundary is supercritical where the curve is solid, subcritical where dashed (see section 3.3). The dot-dashed curves show the asymptotic stability boundaries (12) and (13).

Hopf bifurcation (see section 3.3 below). Note that $\bar{V}_p$ is displayed in units of km^2, normalizing by $\rho_w g$; in other words, what is really plotted is $\bar{V}_p/(\rho_w g)$, the surface area of the lake (see section 2). Similarly, we will plot N in units ~~of metres~~in subsequent figures, normalizing by $\rho_w g$: changes in N plotted then correspond directly to changes in water level in the lake.

In each case shown in figure 2, there is a region of instability above some critical value of $\bar{V}_p$, and for q_{in} in some intermediate range, where larger $\bar{V}_p$ is required and the unstable inflow range is shrunk for shorter flow path lengths L. This is consistent with theoretical results (section 3.1 of the supplementary material) showing that the system is always stable for either sufficiently small or sufficiently large q_{in}, but can be unstable at intermediate inflow rates.

For large enough $\bar{V}_p$ and L, we can determine approximate analytical expressions for the stability boundaries. The lower critical value of q_{in} at which the drainage system first becomes unstable corresponds to the switch from a cavity-like to a channel-like conduit, and this occurs for large L at (see Schoof, 2010, and section 3.1 of the supplementary material)

$$q_{in} = \frac{u_b h_r}{c_1(\alpha - 1)\Psi_0}. \tag{12}$$

The upper critical value of q_{in} at which the system stabilizes again can similarly computed in the limit of large L by omitting the cavity opening term ~~v_0~~ v_o and reducing conduit evolution (2a) to a balance between the first (dissipation-driven melting) and third (creep-closure) terms on the right hand side. This yields

$$\bar{V}_p \sim \frac{q_{in}^{1/\alpha}}{2(\alpha-1)c_1 c_3^{1/\alpha}\Psi_0^{(3\alpha+2)/(2\alpha)}L}. \tag{13}$$

These limiting forms are also shown in figure 2, where we see that they are most accurate for large L as expected.

3.3 Hopf bifurcations and limit cycles

The analysis above has been purely linear, identifying parameter regimes in which steady drainage is unstable. ~~What the analysis does not allow is to say is what happens when the perturbations grow in size to the point where the linearization fails.~~

That linearization will eventually fail where unstable growth is predicted, and a nonlinear model is needed to predict what the instability grows into. A key aspect of many outburst floods is that they are a recurring phenomenon~~; in terms of our~~. For the reduced model (2) with two dynamical degrees of freedom and steady forcing, such recurrence must correspond to a stable periodic oscillation in the absence of time-dependent forcing (see also Kingslake, 2013, for time-dependent forcing that leads to chaotic solutions). As was underlined by Ng (1998) and Fowler (1999), the existence of such a limit cycle does not simply follow from the instability itself: we need to ensure that the evolution away from the steady state leads to bounded growth of the instability once it reaches a finite amplitude~~(as opposed to the infinitesimal perturbation assumed by the linearization in section 3.1). This cannot be done in the confines of the linearization of section 3.1 alone~~.

It is straightforward to demonstrate bounded growth in our model computationally. Figure 3 shows a sample calculation of a periodic solution, with the classical attributes of a outburst flood cycle: effective pressure slowly decreases during the interval between outburst floods, when conduit size is small. This is simply the lake refilling. As N approaches its minimum, the conduit size S starts to grow rapidly, initiating the outburst flood: effective pressure is no longer large enough to keep conduit size small, and enough water can flow to start enlarging the conduit in the runaway growth envisioned by Nye (1976). The lake then drains, rapidly increasing N. Once effective pressure gets large enough, creep closure becomes dominant, causing S to shrink again, and the cycle repeats. By contrast with Ng (1998) and Fowler (1999), key to this periodic behaviour is that the conduit cannot become arbitrarily small between floods, since it is kept open by ice flow over bed roughness.

In fact, S in our model actually starts to increase almost immediately after flood termination, as the refilling of the reservoir leads to decreasing effective pressure allowing the now cavity-like conduit to grow. As explained above, water flow through them will eventually lead to re-initiation of dissipation-driven melting and enlargement of the conduit. By contrast, once the amplitude of the floods has become large enough, conduit size in the pure channel model of (Ng, 1998) and (Fowler, 1999) keeps shrinking during the refilling phase until the effective pressure changes sign to negative, and this underpins the ever-increasing flood amplitudes in their model.

An alternative ~~and possibly preferable way to visualize the periodic solution is in~~ visualization of the physics involved is a phase plane, plotting S against N as the system evolves; a periodic solution then corresponds to a closed orbit. Figure 4 shows

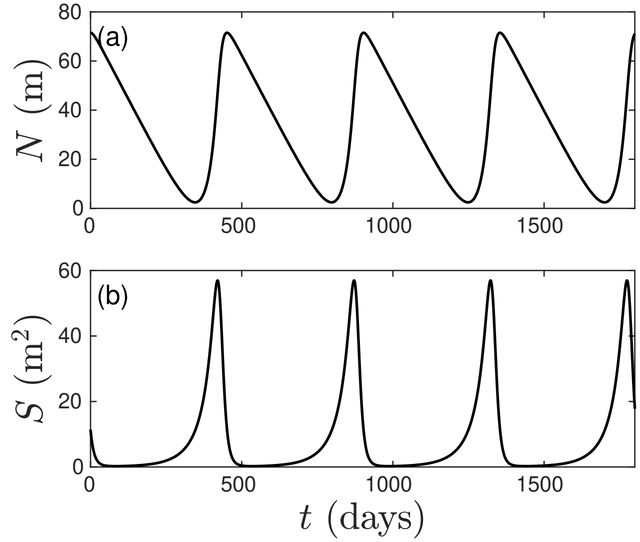

Figure 3. Periodic oscillations in the system (2) with parameter values as in table 1, with V_p = 4 km^2, L = 50 km and q_{in} = 10.9 m^3 s^{-1}.

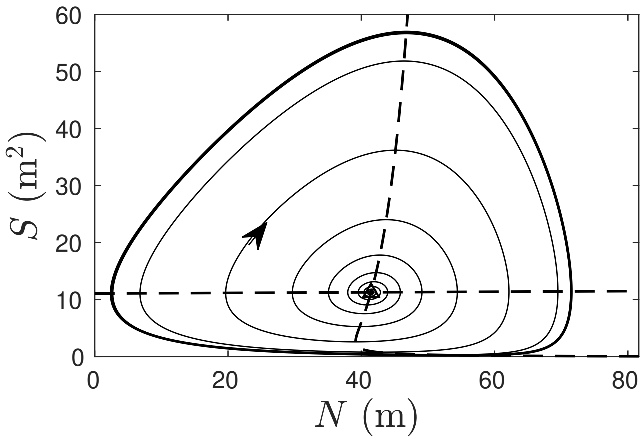

Figure 4. The periodic solution of figure 3 plotted in a phase plane. The thin curve represents a solution that approaches the limit cycle shown as a thick curve as indicated by the arrow~~, and the~~ . The instability of the steady state solution (where the nullclines cross) is clearly visible, since evolution along the thin line is away from that steady state.

the phase plane equivalent of (3), with the dashed lines corresponding to nullclines (the curves on which either $\dot{S} = 0$ or $\dot{N} = 0$. During the refilling phase, the point $(N(t), S(t))$ closely follows the S-nullcline, with S steadily increasing as explained above.

Figures 3 and 4 show only one example~~, and the question remains as to whether closed orbits invariably result from the instability of section 3.1, and how these orbits change as parameter values are changed~~. It is possible to solve directly for such periodic solutions and trace how they change under parameter changes using an arc length continuation method (section 3.4 of the supplementary material), and to determine simultaneously whether the periodic solutions are stable: ~~this is not guaranteed, and~~ some periodic solutions are in fact never attained by forward integration of (2) because they are themselves unstable small perturbations will cause the system to evolve away from them.

As in section 3.2, we focus on how solutions change as the water input to the reservoir q_{in} is varied. Figure 5 shows how the amplitude of oscillations varies with water input for a fixed $L = 50$ km and different storage capacity V_p (treated as independent of N here). In each panel, the coloured curve (black in panel e) shows the minimum and maximum value of N attained for a periodic solution at the corresponding value of q_{in} (i.e., the values of N where the corresponding orbit in the phase plane crosses the N-nullcline). Solid portions of these coloured curves in figure 5 correspond to stable periodic solutions, while dotted portions are unstable periodic orbits. The black curves in figure 5 generally correspond to steady state solutions, plotting N in the steady state against q_{in} (we use black for steady states and oscillatory solutions in panel e, but these are easily distinguished by comparing the plots in panel e with those in the remaining panels). Again, solid black lines are stable steady states and dashed black lines are unstable steady states. Each panel in figure 5 corresponds to a different storage capacity V_p and therefore represents a horizontal slice through panel figure 2; V_p decreases from panel a to panel d.

In all cases, an unstable steady state corresponds to a limit cycle. As q_{in} crosses the lower critical value at which instability first occurs (associated with the change in the steady state conduit from cavity- to channel-like behaviour), a limit cycle of small amplitude is formed, with that amplitude growing progressively as q_{in} increases~~. This is consistent with a supercritical Hopf bifurcation: in general, the change from stability to instability in our model corresponds to the eigenvalue λ in attaining the purely imaginary value $\pm i\sqrt{a_2}$, and a standard result from the theory of dynamical systems is that a local, small-amplitude periodic solution exists near the bifurcation (Wiggins, 2003). What is non-trivial to determine *a priori* is whether that periodic oscillation exists on the stable or unstable side of the bifurcation (in this, for values of q_{in} less than or larger than the critical value, respectively). The stability of the periodic solution complements that of the corresponding steady state: if the periodic solution appears on the side of the bifurcation where the steady statehas become unstable (a case termed~~ , and the emerging limit cycle is an oscillation around an unstable steady state. This is consistent with a supercritical Hopf bifurcation ~~), then the periodic solution is stable. This is the case for the lower critical value of q_{in} in all panels of figure 5~~(Wiggins, 2003).

As the upper critical value of q_{in} is approached, the stability analysis of section 3.1 predicts that the steady state returns to stability. As this happens, we see that the amplitude of oscillations only shrinks continuously back to zero for the smallest value of $\bar{V}_p$ considered (in which case the upper critical value corresponds to another supercritical Hopf bifurcation). In panels a-c, the amplitude of stable periodic solutions continues to increase until values of q_{in} near the upper critical value are reached. The amplitude then decreases slightly with a further ~~increases~~ increase in q_{in}, before the periodic solution ceases to exist at a third critical value of q_{in} that is larger than the threshold at which the steady state has become stable again, as shown in more ~~detal~~ detail in the inset in panel b. (In technical terms, this is known as a saddle-node bifurcation of the Poincaré map of the dynamical system (2), see Wiggins (2003).)

Where this abrupt disappearance of the stable periodic solution occurs, it is ~~generally accompanied by the existence of~~ accompanied by an unstable periodic solution near the upper critical value of q_{in} ~~at which the steady state becomes stable again~~

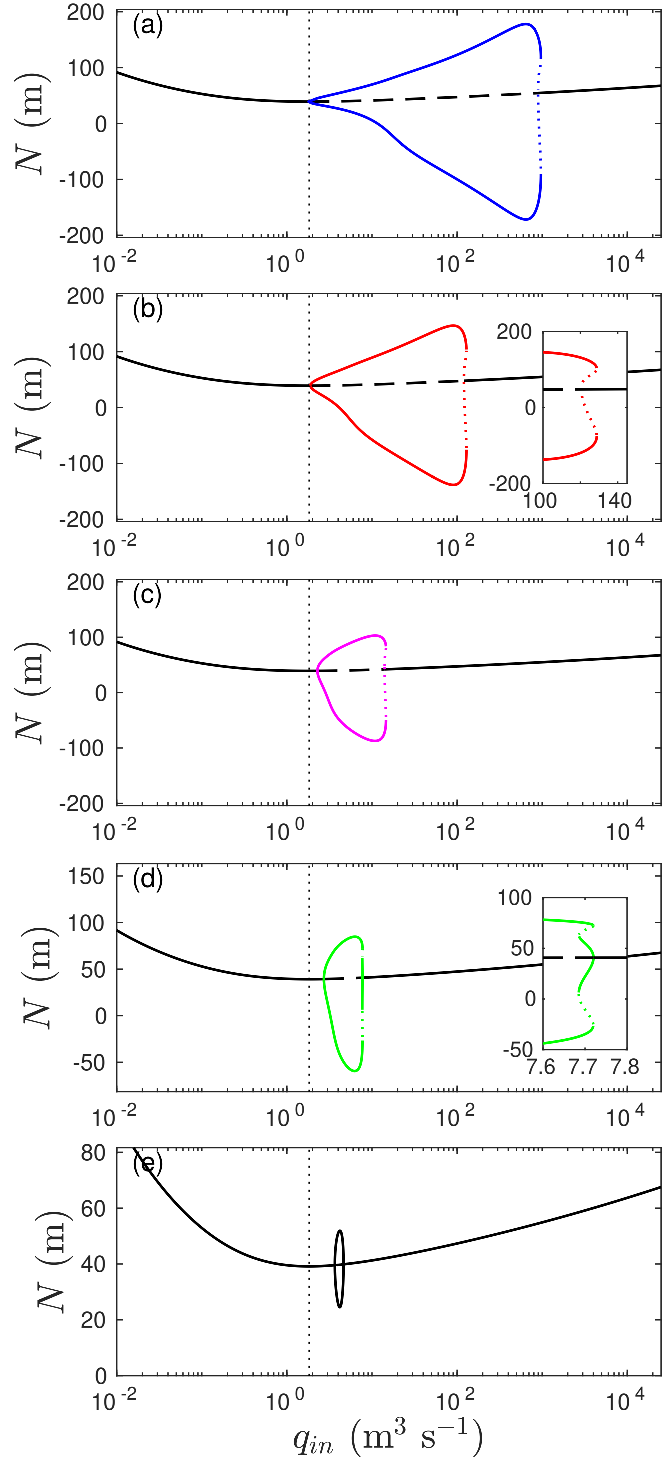

Figure 5. Bifurcation diagrams for $L = 50$ km, $V_p = 4$ (a), 0.8 (b), 0.16 (c), 0.112 (d), 0.094 (e) km^2. Plotted are stable (solid black) and unstable (dashed black) steady state N, minimum and maximum N attained in stable (solid coloured) and unstable (~~solid dashed~~dotted coloured) periodic solutions. The ~~insest~~ insets in (b) and (d) show details, while the vertical dotted line marks the value of q_{in} at which the conduit becomes channel-like.

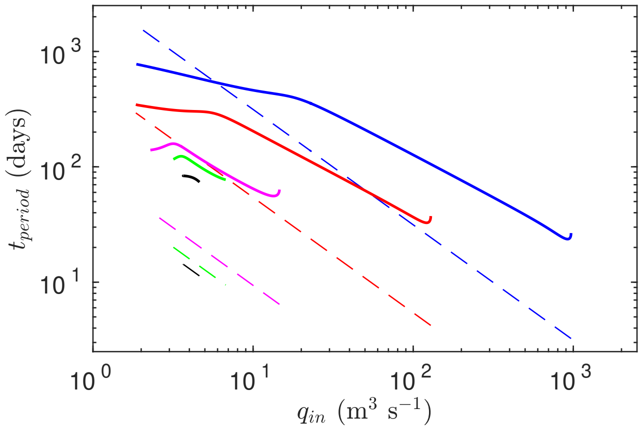

Figure 6. The period of stable periodic solutions where they exist for the same parameter values as in figure 5. The colour scheme is the same as for the coloured curves in figure 5, where only the stable portion of the periodic solution curves is shown. The dashed line here represent the asymptotic solution for the period of oscillation, (14)

~~(see in particular the inset in panel b). That critical value generally corresponds to a so-called~~ , corresponding to a subcritical Hopf bifurcation~~(as a counterpart to its supercritical cousin that we encountered above)~~. Panel d (see inset) represents an exotic exception to this, where there are two stable periodic solutions for a small interval of q_{in}.

The nature of the Hopf bifurcations at the stability boundaries in figure 2 can be determined more quickly than by the numerical continuation method used above, using a weakly nonlinear stability analysis ~~in the vicinity of the stability boundary~~ (see section 3.3 of the supplementary material). This allows us to map out in figure 2 where that boundary corresponds to a supercritical Hopf bifurcation, in whose vicinity a small-amplitude stable periodic solution emerges (stability ~~boundary indicated~~ boundaries shown as a solid curve), and where the Hopf bifurcation is subcritical and the ~~stable periodic solution has a large amplitude~~small-amplitude periodic solution is unstable (dashed stability boundary). As in the few samples shown in figure 5, we see that the lower critical value of q_{in} is always supercritical, while the upper critical value is generally subcritical, except at low V_p.

~~In practical terms, figure 5 shows that for every unstable steady state, there is a corresponding stable periodic solution, so the system evolves away from the steady state into an oscillation of finite amplitude as shown in figures 3 and 4. In practice, we may wish to know not only how the amplitude of oscillations(~~We can also compute the period of the the drainage oscillations, that is, ~~of variations in N during the flood cycle) depends on water supply to the reservoir, but also what the recurrence period of the floodsis~~the recurrence interval of floods. Figure 6 shows the period of the stable periodic solutions shown in figure 5 as a function of water supply rate q_{in}, using the same colouring scheme to distinguish different values of V_p. Perhaps unsurprisingly, large inflow rates q_{in} correspond to more rapid flood cycles, and large reservoir volumes correspond to floods repeating more slowly, albeit with a larger amplitude.

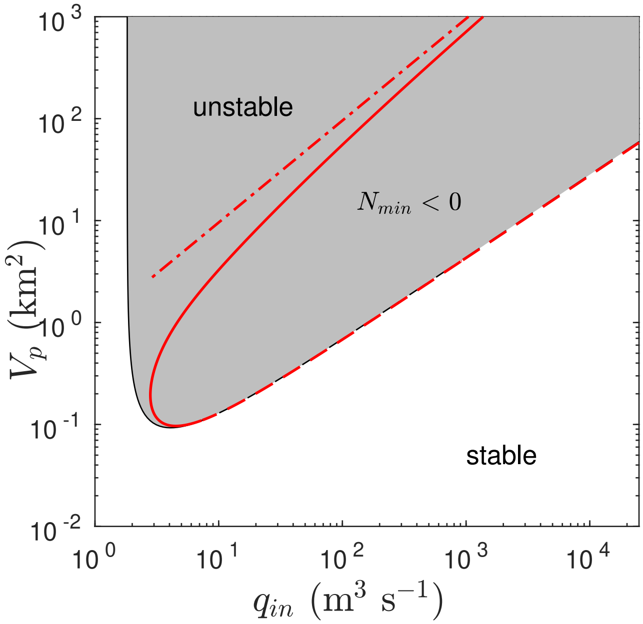

Figure 7. The stability boundary of figure 2 for $L = 50$ km plotted in black, with the region of parameter space in which the steady state is unstable shown in grey. The red curve is set of parameter combinations $(q_{in}, V_p))$ for which there is a periodic solution that reaches a minimum effective pressure of $N = 0$. The minimum effective pressure becomes negative inside the region delineated by the red curve, as well as in a small strip to the right of the right-hand branch of the red curve: this is the region where the stable limit cycles shown for instance in the inset of figure 5(b) coexist with stable steady state solutions, and exist for larger q_{in} than the periodic orbit that reaches a minimum of $N = 0$ near the upper critical value of q_{in}. The dot-dashed curve is the asymptotic formula (15) for the threshold value of q_{in} at which negative N is first attained.

There is a significant caveat here: in many cases, the limit cycle solutions we have computed predict that N becomes negative during the cycle of reservoir filling and draining. This is not something that the ~~model , or indeed its spatially extended counterpart , were intended to capture~~models are designed for. Instead of the ~~creep closure of the conduit $v_c(S,N)$ simply becoming negative and the conduit ‘creep-opening’~~conduit expanding through creep, as occurs in (1) and (2) when $N < 0$, we expect that the glacier should instead detach from the bed and a sheet flow of water initiates the outburst flood in this case; prototypes of models for this ~~eventuality~~ behaviour can be found in Schoof et al. (2012), Hewitt et al. (2012) and Tsai and Rice (2010).

It is important to know where in parameter space the outburst flood mechanism will change away from conduit growth due to cavities becoming channel-like at the end of the refilling phase, and instead involve water separating ice from bed at vanishing effective pressure. The boundary between these two regimes should correspond to the location in parameter space where the minimum effective pressure during the flood cycle is zero. Figure 7 shows that parameter regime boundary, computed numerically (section 3.4 of the supplementary material) and superimposed on the stability boundary plot of figure 2. Clearly, sheet-flow-initiated floods (in which the glacier starts to float at the start of the outburst flood) are favoured at high water input rates and smaller reservoir volumes, where the conduit is less able to adjust to the rapid refilling of the reservoir.

3.4 Asymptotic solutions

We can also address limit cycle solutions through asymptotic methods in some parametric limits in our model. The most relevant limit for 'real' glacier-dammed lakes is likely to be that of a relatively large reservoir that is filled relatively slowly, but where water supply is not so small as to allow the conduit to be cavity-like in steady state (in which case the reservoir would be drained steadily, without a flood cycle): This is the case of large V_p and moderate but not small q_{in}, and is described in appendix A and, in detail, in section 4 of the supplementary material. This region of parameter space lies in the upper left of the unstable region of figure 7, between the near-vertical solid black line and the solid red curve.

In brief, the asymptotic solution confirms that there is a periodic flood cycle that the system very quickly settles into, and that the flood cycle consists of three distinct stages. During the main flood stage, the evolution of the conduit is rapid and dominated by dissipation-driven melt $c_1 q \Psi$ and creep closure $v_c(S,N)$ in (2a), and water input q_{in} to the lake is much smaller than outflow q. This is followed by a long refilling phase in which the conduit has shrunk dramatically and therefore behaves as a cavity. Its size is dictated by a quasi-equilibrium in which the cavity opening term $v_o(S)$ balances the creep closure term $v_c(S,N)$, leading to a slow opening of the conduit as the reservoir fills and N consequently dropped. Outflow q from the reservoir during this phase is insignificant, and the mass balance of the lake is dominated by inflow q_{in}. The refilling phase is terminated by a flood initiation phase whose length is intermediate between the main refilling phase and the rapid flood phase. During this initiation phase, the reservoir is still filling, but the conduit has enlarged sufficiently that dissipation-driven wall melting $c_1 q \Psi$ starts to be significant.

Qualitatively, this solution is illustrated by the limit cycle in figures 3 and 4 and panel a of figure 9. The asymptotic solution only becomes quantitatively accurate when V_p is large enough to be physically unrealistic for real glacier-dammed lakes (section 4.4 of the supplementary material); this is because the relatively large exponent $n = 3$ in Glen's law makes the creep closure term quite sensitive to changes in N when N is small, and the initiation phase of the floods is affected significantly by this.

One of the predictions of the asymptotic solution is that the amplitude of effective pressure oscillations should be insensitive to refilling rate q_{in}, except close to the Hopf bifurcation at which oscillations are initiated. As a result, the flood recurrence period (which is essentially the time taken to refill the reservoir in the limit where reservoir drainage is fast) should simply be inversely proportional to q_{in}. Specifically, the result states that

$$t_{period} \sim \equiv \tilde{N}_f c_1^{\alpha/(n+1-\alpha)} c_2^{-1/(n+1-\alpha)} c_3^{1/(n+1-\alpha)}$$

$$\times \Psi_0^{(1+2\alpha)/[2(n+1-\alpha)]} V_p^{n/(n+1-\alpha)} q_{in}^{-1}, \tag{14}$$

where $\tilde{N}_f$ is a dimensionless constant, with a value of 1.44 for the parameters of α and n chosen here, in the limit of a large flow path length L (section 4.1 of the supplementary material). This asymptotic formula is overlaid onto the numerically computed periods in figure 6; it should be clear that the asymptotic formula performs poorly for the relatively moderate values of V_p used here. This should not be a surprise since figure 5 demonstrates that, for the same values of V_p, the amplitude of oscillations is in fact sensitive to the inflow rate q_{in}.

The same asymptotic solution also provides an estimate for the inflow rate q_{in} at which zero effective pressure is first reached during the flood cycle, and our model ceases to be physically realistic as described above. The estimate is given by an analysis of the flood initiation phase (section 4.4 of the supplementary material) as

$$q_{in} \sim \gamma_c c_1^{(n+1)/(\alpha n)} c_2^{-1/n} c_3^{(n+1)/(\alpha n)}$$

$$\times \quad \Psi_0^{3(n+1)/(2\alpha n)} (u_b h_r)^{-(\alpha-1)(n+1)/(\alpha n)} V_p, \qquad (15)$$

~~This~~ where $\gamma_c \approx 0.25$ for the values of α and n chosen here. This is critical inflow rate as a function of V_p is superimposed on figure 7. The formula above is in general an underestimate of the value of q_{in} at which zero effective pressures are reached, but clearly gives the correct scaling of how that value relates to V_p.

We are also able to construct a second asymptotic solution for the opposite case of a large water supply rate q_{in} (section 5 of the supplementary material). This predicts rapid oscillations in the reservoir level (or effective pressure) whose amplitude slowly evolves to a steady value. These rapid oscillations however invariably involve effective pressure changing from negative to positive~~: at leading order the growth and shrinkage of the conduit is controlled by the creep ‘closure’ term, which becomes a creep opening term during about half of the cycle as in panel b of figure 9, and dissipative melting is a higher order correction that ultimately dictates the amplitude that the rapid oscillations settle into over time~~. As discussed above, such solutions are clearly not physically viable, and the model must be amended to take account of ice-bed separation~~; the asymptotic solution here serves merely to confirm that negative effective pressures are a robust prediction of our model for these large inflow rates, when the possibility of ice-bed separation is not taken into account~~.

4 The spatially extended model

The analysis ~~described~~ above provides a comprehensive picture of the ~~qualitative attributes of the~~ lumped model (2). Here, we consider how well that lumped model represents the behaviour of ~~the~~ its more complete, spatially-extended counterpart (1). We begin by recreating the stability boundary diagrams of figure 2. The method by which the latter were computed explicitly (sections 3.1 and 3.2 of the supplementary material) cannot be applied directly to the extended model (1), since we have no closed-form solution to a linearized version of the model analogous to (9). Instead, we grid the $(q_{in}, \bar{V}_p)$ parameter space used in figure 2 finely. For each $(q_{in}, \bar{V}_p)$ pair, we discretize (1), compute steady states and perform a linear stability analysis numerically as described in ~~Schoof et al. (2014)~~Schoof et al. (2014, appendix B); this allows us to delineate regions

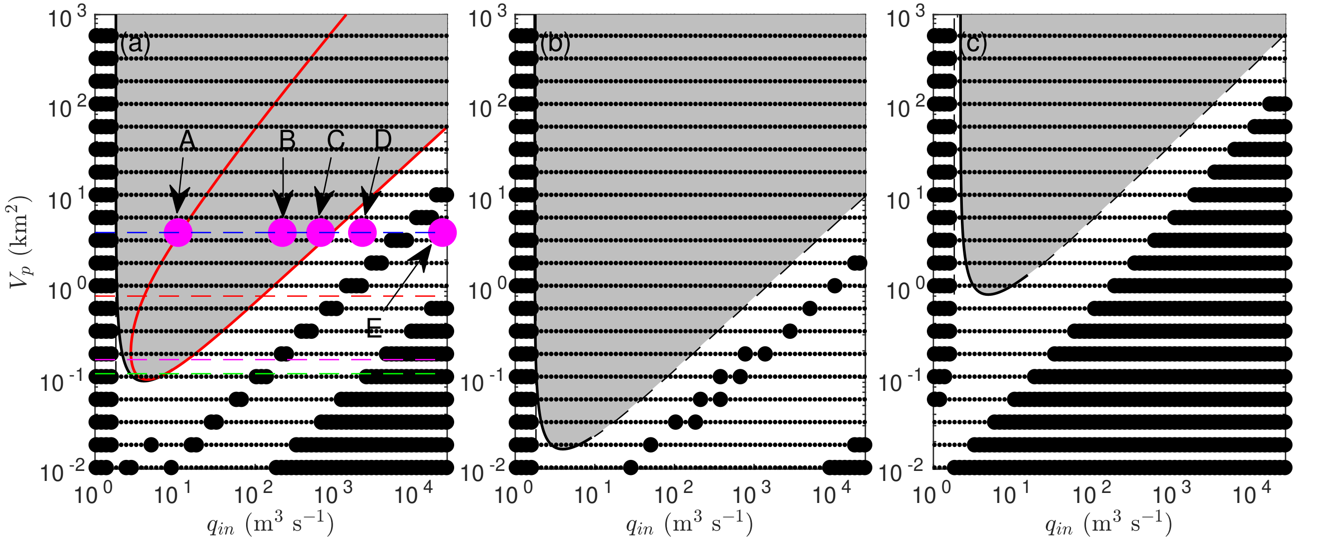

Figure 8. The stability boundaries of figure 2 with L =50 km (a), 250 km (b), 10 km (c) plotted ~~in~~ as black lines; unstable regions in the lumped model (2) are marked in grey. Solid black circles indicate parameter combinations for which the steady state solution to the full model (1) is stable, small dots indicate that the full model is unstable. The magenta markers in panel (a) indicate parameter values for which the evolution of the full, spatially extended model has been computed up to finite amplitude (figures 9–11). The red curve in panel (a) is the same boundary as in figure 7 indicating where negative effective pressures are encountered in the lumped model. The dashed, coloured horizontal lines in (a) show the values of V_p used in figure 8, using the same coloured scheme as the periodic orbits in the latter figure.

of instability, although in a less sophisticated way. We adapt that method for the transient nonlinear calculations in figures 9–11 by solving equations (B1)–(B2) of the same appendix in Schoof et al. (2014) using a backward Euler step.

Results are shown in figure 8. It is clear that the lumped model consistently underestimates the range of parameter values over which steady drainage is unstable. The onset of instability at the transition from cavity- to channel-like conduit behaviour ~~appears to remain robust except in the case of~~ remains robust except for small domain lengths L (panel c), ~~for which~~ where the system appears to be unstable for combinations of small storage capacities $\bar{V}_p$ and inflow rates q_{in}. ~~Where the~~ The lumped model typically underestimates instability ~~is~~ at low storage capacities, and at large flow path lengths L. The latter is particularly significant since we have previously attributed stabilization to the effect incipient reservoir drainage reducing the hydraulic gradient along the flow path and therefore reducing flow through the conduit.

While stabilization at large water input rates q_{in} does eventually occur, this ~~occurs~~ happens at values that can be several orders of magnitude larger than ~~those~~ predicted by the lumped model, especially for ~~the case of a~~ large flow path lengths L. Furthermore, for a given storage capacity $\bar{V}_p$, the lumped model predicts that there is a single interval of inflow rate values q_{in} over which instability occurs. The spatially extended model by contrast has two or more such intervals for most values of $\bar{V}_p$, with a narrow region of stability between (the diagonal bands of solid black diamonds in panels a and b of figure 8).

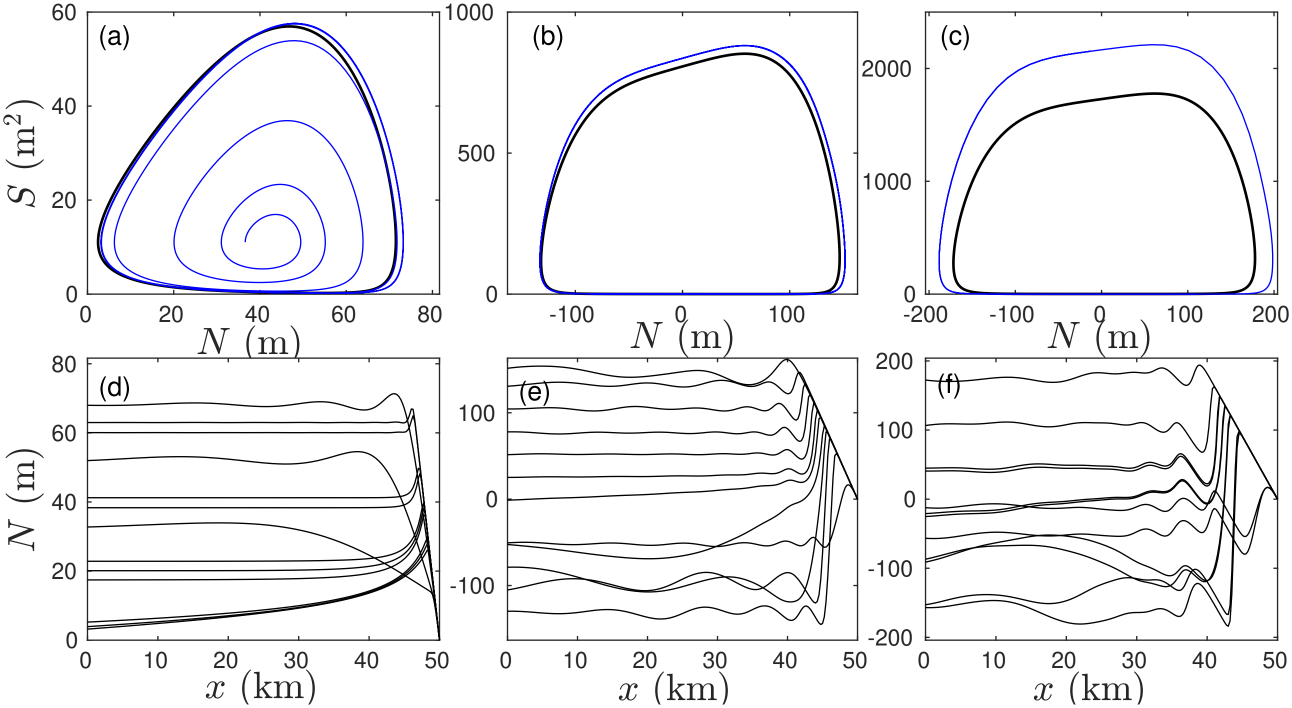

Figure 9. Each column represents one of the magenta markers in figure 8a. For each column, $V_p = 4$ km^2 and $L = 50$ km, while $q_{in} =$ 10.9 m^3 s^{-1} (panels a, d, marker A in figure 8a.), 218.4 m^3 s^{-1} (panels b, e, marker B), 655 m^3 s^{-1} (panels c, f, marker C). The top row (panels a-c) displays phase planes of $(N(0,t), S(0,t))$ for the full model (1) (blue) superimposed on the phase plots for the lumped model (2) (black). The bottom row shows snapshots of $N(x,t)$ against x for periodic solutions of the full model at intervals of 93 days (panel d) 5.5 days (e) and 4.6 days (f).

To understand this discrepancy better. we have solved for the nonlinear evolution of the draiange system as described by the spatially extended model (1) for the parameter values indicated by magenta circles in panel a of figure 8.

The ~~there~~ three smallest values of q_{in} all correspond to unstable steady states in the lumped model (2), and we can compare spatially extended and lumped solutions. Figure 9 shows results. While the spatially extended solution is an infinite-dimensional dynamical system and strictly speaking cannot be visualized using a phase plane, we can overlay plots of conduit size $S(0,t)$ at the upstream end of the conduit against effective pressure $N(0,t)$ at the same location onto the S–N phase plot for the lumped model. This is shown in the top row of panels. In all three cases, the extended models settles into a limit cycle, and for small and intermediate q_{in}, we find good agreement between extended and lumped models~~; this~~. This only breaks down as we approach the upper critical value of q_{in} at which the lumped model stabilizes~~is approached~~.

The lower row of panels in figure 9 shows snapshots of $N(x,t)$ against x for the correspnding limit cycle shown in the top panel of the same column. For the smaller two values of q_{in}, we see that pressure gradients $\partial N/\partial x$ are ~~in general~~ moderate away from the glacier terminus and therefore do not contribute significantly to the hydraulic gradient Ψ~~: this~~. This was the basis for the reduction of the spatially extended model (1) to the lumped form (2), and explains the good agreement. For larger q_{in}, effective pressure $N(x,t)$ along the conduit starts to develop wave-like structures, causing the approximation of negligible pressure gradients to break down, and the discrepancy between lumped and extended model grows.

The extended model remains unstable beyond the stability boundary ~~model~~ of the lumped model, but our numerical solutions no longer support the conclusion that the system necessarily settles into a limit cycle. Figure 10 shows the evolution of the system for $V_p = 408$ m^3 Pa^{-1} s^{-1} (a lake with a surface area of 4 km^2) and $q_{in} = 2.18 \times 10^3$ m^3 s^{-1} as in the magenta marker labelled 'D' in figure 8a (the fact that this is an inflow rate of biblical proportions is not as relevant as it may at first seem, as we discuss in section 5 below). Panels a and b show the growth of rapid oscillations in the maxima of N and S along the flow path over time, but without the growth becoming bounded. Panels c and d show snapshots over the last two oscillations in the computation. Note that the vertical scale in panel c is logarithmic. What we see is that conduit size develops an aneurysm-like, massively enlarged feature near the terminus, blocked by a narrow constriction that requires an extremely large pressure gradient to overcome~~(note that panel c uses a logarithmic verticla scale)~~. The simulation eventually terminates when the solver fails to converge, and it is unclear whether this is merely a computational problem or indicates that the true continuum solution becomes pathological or ceases to exist.

For even larger q_{in}, a narrow band of stable values is passed and a different instability ensues as shown in figure 11. This instability appears to lead to bounded growth, again marked by rapid oscillations whose amplitude ~~slowly grows~~ first grows and then saturates, and significant pressure gradients along the flow path.

Once again, a key observation is that the spatially extended model predicts negative effective pressures long before any kind of limit cycle is reached for all but one of the solutions shown, and that the discrepancies between lumped and spatially extended models occur only where this is the case. Moreover, they occur for water supply rates to the reservoir that far exceed values that would be plausible for typical glacier-dammed lake systems: the lumped model appears to be robust for the latter~~, including its prediction~~. This includes the prediction by the lumped system of where overpressurization of the drainage system ~~with negative N~~ ($N < 0$ in the limit cycle) first occurs.

~~Nonetheless, as we will discuss shortly, this~~ This does not render some of the more exotic instabilities shown in figures 9–11 irrelevant: The parameter values we have chosen here were deliberately chosen to reflect typically large reservoirs dammed by low-angled and large glaciers. Much smaller reservoirs in shorter, more steeply-angled glaciers are liable to give rise to some of ~~the instabilitiesthat~~ these exotic instabilities, even though they lie outside the reach of standard glacier-dammed lakes.

5 Discussion

5.1 Glacier-dammed lakes and small reservoirs

We have shown that a drainage system that switches between cavity-like and channel-like behaviour spontaneously is capable of supporting periodic outburst floods from a glacier-dammed reservoir. At issue here is the initiation of the flood: as discussed by Ng (1998) and Fowler (1999), if the conduit consists purely of a Röthlisberger-type channel, kept open only by melting

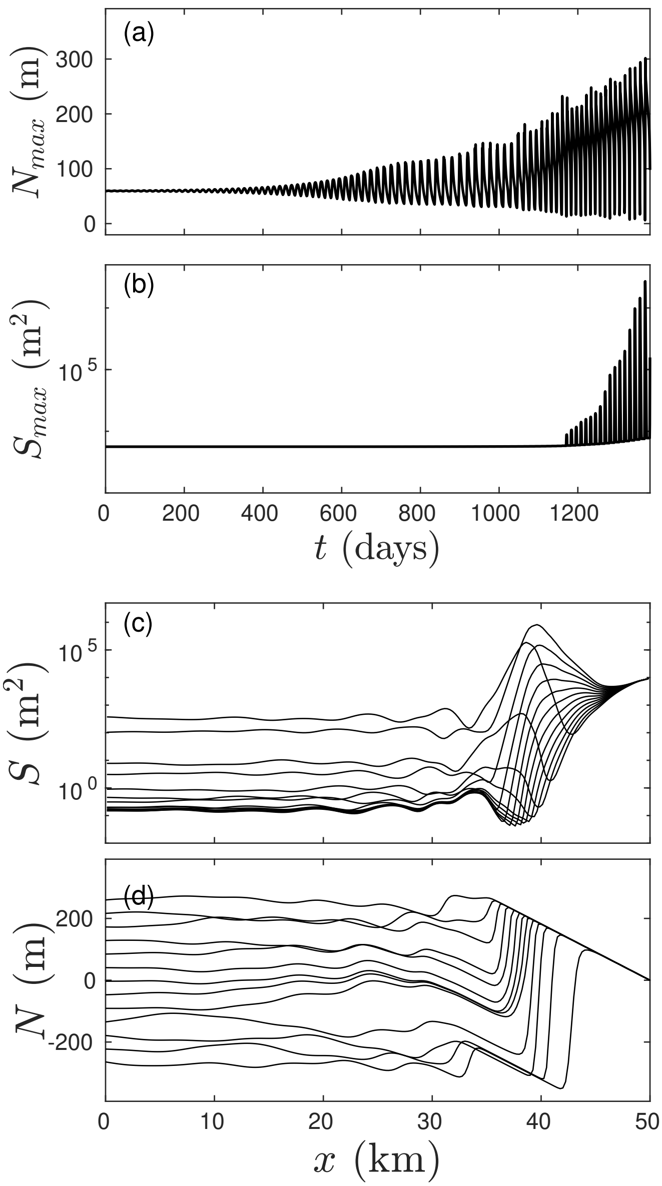

Figure 10. Finite-amplitude evolution of the instability for the magenta dot marked 'D' in figure 8a. (a): the maximum of $N(x,t)$ along the flow path plotted against time t. (b): the maximum of $S(x,t)$ with respect to x plotted against t (c): snapshots of $S(x,t)$ at intervals of 0.93 days towards the end of the simulation (d): snapshots of $N(x,t)$ at the same points in time as in (c). Note the logarithmic vertical scale in (c).

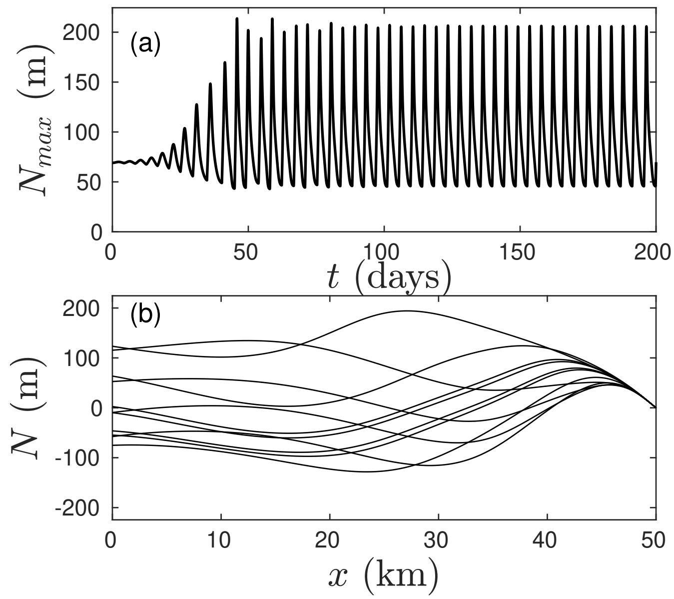

Figure 11. Finite-amplitude evolution of the instability for the magenta dot marked 'E' in figure 8a. (a): the maximum of $N(x,t)$ along the flow path plotted against time t. (b): snapshots of $N(x,t)$ at the same points in time as in (c).

due to heat dissipated in the turbulent flow of water through the channel, then the initiation of floods can be delayed more and more, without a limit cycle emerging. Both of these authors proposed that englacially-routed water supplied to the channel should keep it open at a minimum size, and the migration of the flow divide within the channel as the lake fills then becomes the control on flood initiation. Our theory differs from theirs in the sense that no englacial water supply is necessary and the lake can continuously leak water through a linked-cavity-type drainage system that spontaneously becomes 'channel-like' and undergoes runaway enlargement through dissipation-driven melting as the lake fills.

At the heart of the oscillatory behaviour of glacier-dammed lakes is ~~exactly that instability of a channel~~that runaway growth, or instability, of a channel-like conduit, which prevents steady discharge of the water supplied to the reservoir (Nye, 1976; Ng, 1998; Fowler, 1999). As described in Schoof et al. (2014) and section 3.1 above, this instability does not occur when the conduit draining the reservoir acts as a set of linked cavities: when these are capable of steadily draining the water supplied to the reservoir, no outburst floods occur.

By contrast, for typical glacier-dammed lakes, ~~in which~~ moderate inflows q_{in} exceed the drainage capacity of a linked cavity system but the ~~the~~ large storage capacity V_p ensures that the lake fills slowly compared with the time scale over which a basal conduit can evolve~~, the~~. The cycle of filling and draining then follows the characteristic sequence of a long filling period in which outflow from the lake is negligible, followed by a brief onset period in which the conduit starts to experience significant melt-driven enlargement but the lake level continues to rise, and an even shorter outburst flood in which inflow to the lake is dwarfed by drainage through the subglacial conduit. For a given lake size V_p, larger inflow rates will slowly increase the amplitude of the lake level fluctuation during the flood cycle (figure 5), while significantly shortening the length of the flood cycle (figure 6).

As in Ng (1998) and Fowler (1999), our flood initiation mechanism can also be too slow to respond to water input and lead to our models (1) and (2) predicting negative effective pressures ~~in the reservoir~~at the upstream end of the conduit: in that case, the glacier ought to float and flood initiation is likely to take the form a sheet flow that subsequently channelizes, as explored in Flowers et al. (2004), Schoof et al. (2012) and Hewitt et al. (2012). This effect is not included in our models here, but we are at least able to give an ~~asymptotically valid (~~approximate expression (valid in the limit of very large reservoir sizes V_p) ~~criterion~~ for the water supply ~~rate~~ q_{in} at which the change in flood initiation mechanism occurs (switching between between unstable conduit growth and partial glacier flotation ~~occurs (~~and a sheet flow); this is equation (15)~~)~~. Adapting our model to account for this alternative flood initiation mechanism is the obvious next step to take. One straightforward adaptation of the lumped model to this effect, based on the model in Hewitt et al. (2012) is the following: if suppose that lake level cannot rise beyond the point at which $N = 0$, then the opening of an ice-bed gap must be such as to permit outflow balancing inflow to the lake at that point, which forces us to alter the constitutive relation for q once $N = 0$. We can similarly build the possibility of flood termination due to the lake running dry at an upper bound on effective pressure at $N = N_{max} = \rho_i g(s(0) - b(0))$ into the model. Once the lake is fully empty, discharge q in the conduit will be the lesser of inflow into the conduit q_{in} (corresponding to a partially filled conduit) and the discharge the conduit would carry if it were completely filled with water. The appropriately adapted lumped model (2) is

$$\dot{S} = c_1 q \Psi + v_o(S) - v_c(S, N) \tag{16a}$$

$$-V_p \dot{N} = q_{in} - \tilde{q} \tag{16b}$$

$$\tilde{q} = \begin{cases} \max(q(S,\Psi), q_{in}) & \text{if } N = 0, \\ q(S,\Psi) & \text{if } 0 < N < N_{max}, \\ \min(q(S,\Psi), q_{in}) & \text{if } N = N_{max}, \end{cases} \tag{16c}$$

$$\Psi = \Psi_0 - N/L. \tag{16d}$$

The equivalent for the spatially-extended model (1) is the channel model in Hewitt et al. (2012), with a reservoir at the upstream end. Importantly, the stability of steady state solutions as in sections 3.1 and 4 is unchanged by these adaptations unless the steady state corresponds to an empty lake basin and a partially filled conduit, in which case the conduit is presumably stable since discharge in the conduit does not increase when its size is increased. The upper and lower bounds on flux introduced in (16) only affects the nonlinear dynamics of the system, once the amplitude gets large enough. We leave the nonlinear analysis of the adapted model to future work.

We have also investigated a mechanism previously identified in Schoof et al. (2014), by which flow out of a reservoir can be stabilized when the storage capacity in the reservoir is relatively small, so that typical drainage rates (comparable to the inflow rate q_{in}) lead to adjustment of effective pressure in the ~~lake~~ reservoir much faster than the conduit can evolve due to

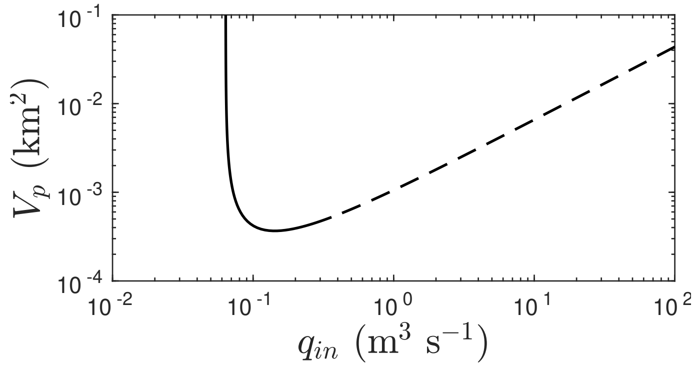

Figure 12. Stability boundary plotted in the same way as in figure 2, but with $\Psi_0 = 1630$ Pa m^{-1}, $u_b h_r = 1.05 \times 10^{-07}$ m^2 s^{-1}, and we put $L = 5000$ m.

wall melting. The rapid response of reservoir water levels can have a large enough effect on hydraulic gradients to stabilize the flow. This is true at least for a short enough drainage system. Importantly, this happens at water throughput rates that are completely unrealistic for typical, large glacier lakes (and even these rates are underestimated by a simplified, 'lumped' model as described in section 4).

The reason why this stabilization mechanism is relevant is that it may explain why much smaller water reservoirs that are typically not recognized as lakes but nonetheless provide storage capacity (such as large moulins) do not invariably generate outburst-flood type behaviour: the stability diagram of figure 2 was generated for parameter values chosen to represent large lakes dammed by a long glacier with a small surface slope, but can easily be rescaled to represent other glacier geometries, reservoir storage capacities and water throughput. The relevant scaling is explored in section 2 of the supplementary material. Here, we confine ourselves to a simple numerical demonstration: figure 12 recalculates the stability boundary for steeper, shorter valley glacier with a relatively small reservoir sizes. We use the same parameter values as in table 1, except for Ψ_0, $u_b h_r$ and$L$. The 'upper' stability boundary (the larger critical value of q_{in} where the lumped model (2) becomes stable again) is much more accessible for reasonable flow rates, and some of the more exotic manifestations of instability explored in section 2.1 are more likely to be within reach of typical water input rates.

With this as motivation, we also briefly discuss a second ~~instability. This~~ 'exotic' instability, which diverges from the classical picture of a glacier outburst flood in that it does not require channel-like behaviour but can occur in a purely cavity-based drainage system. ~~We analyze this instability briefly below, since the work already done in the present paper provides the necessary framework~~Much of the necessary groundwork is already in place from the analysis in section 3.1.

5.2 Distributed storage: a modification of Nye's instability

In section 2, we formulated a model for a single reservoir, for which Nye's instability predicts the onset of oscillations when the conduit becomes channel-like. In Schoof et al. (2014, section 5), the case of spatially spread-out storage capacity, for instance in the form of numerous basal crevasses arrayed along the flow path, was considered in addition to a single reservoir~~,~~ . A numerical linear stability analysis was used to demonstrate that instability can also occur for cavity-like conduits (which are generally stable for a single reservoir) for the case of such distributed water storage. As in the case of a single reservoir, ~~Schoof et al. (2012)~~ Schoof et al. (2014) find that there is a finite range of values q_{in} for which instability then occurs, only that this extends to lower values of q_{in}, where the conduit is cavity-like.

Here we ~~build on the analysis in~~ sketch how to build on section 3.1 to shed further light on ~~their resultts. The model used in Schoof et al. (2012) was~~ the results in Schoof et al. (2014), which were built on the model

$$\frac{\partial v(N)}{\partial t} + \cancel{\frac{\partial q}{\partial x}} \frac{\partial \tilde{q}}{\partial x} = 0, \tag{17a}$$

$$\frac{\partial S}{\partial t} = c_1 q \Psi + v_o(S) - v_c(S,N), \tag{17b}$$

$$\cancel{q}\tilde{q} = q(S,\Psi), \tag{17c}$$

$$\Psi = \Psi_0 + \frac{\partial N}{\partial x}. \tag{17d}$$

~~where~~ Here $v(N)$ is a ~~*decreasing*~~ decreasing function of N that describes storage of water per unit length of the conduit, and the rest of the notation used replicates that of the single-reservoir model (1). Below, we will denote by

$$v_p = -\frac{\mathrm{d}v}{\mathrm{d}N}, \tag{18}$$

the equivalent of V_p in (1). This model also requires boundary conditions; an obvious choice is a prescribed flux $q = q_{in}$ at the inflow $x = 0$, with no actual reservoir there, and again $N = 0$ at the terminus $x = L$.

~~In order to take advantage of the work already done in section 3.1, we adopt an analytical approach to understanding the instabilities of this modified drainage model, complementing the numerical stability analysis of Schoof et al. (2014) and providing additional insight into the feedbacks involved.~~

With a finite domain size and the proposed boundary conditions above, a steady state solution to (17) will in general have spatial structure as in Schoof et al. (2014), and a linearization around that steady state, analogous to section 3.1 will lead to a boundary value problem ~~with non-constant coefficients~~ that is not amenable to a closed-form solution. To avoid this, we concentrate here on shorter length scales, and assume that we can use (17) with periodic boundary conditions at these scales. This yields a spatially uniform steady state solution defined implicitly by

$$c_1 \bar{q}\bar{\Psi} + v_o(\bar{S}) - v_c(\bar{S},\bar{N}),$$

$$\bar{q} = q(\bar{S},\bar{\Psi}),$$

$$\bar{\Psi} = \Psi_0.$$

~~with a constraint on $\bar{N}$ through the prescribed volume of water in the system(which is preserved due to the assumed periodic boundary conditions). Note that this~~ (7a) combined with $q_{in} = q(\bar{S},\bar{\Psi})$ and $\bar{\Psi} = \Psi_0$, where q_{in} is the prescribed flux through the system. This is closely analogous to (7) but simpler, as the hydraulic gradient here does not contain the gradient term retained in (7c).

Linearizing as $N = \bar{N} + N' \exp(ikx + \lambda t)$, $S = \bar{S} + S' \exp(ikx + \lambda t)$, we find an analogue to the problem (8) as

$$-\bar{v}_p \lambda N' + ikq_S S' - k^2 q_\Psi N' = 0, \tag{19a}$$

$$\lambda S' - (c_1 q_S \bar{\Psi} + v_{o,S} S' - v_{c,S}) S' - \left[ikc_1(q_\Psi \bar{\Psi} + \bar{q}) - v_{c,N}\right] N' = \cancel{0.}0, \tag{19b}$$

using the same notation as in section 3.1. The eigenvalue λ ~~satisfies a quadratic~~ is again given by an equation of the form ~~, with solution~~ (9), but now with a_1 and a_2 given by

$$a_1 = (c_1 q_S \bar{\Psi} + v_{o,S} - v_{c,S}) - \bar{v}_p^{-1} k^2 q_\Psi \tag{20a}$$

$$a_2 = \bar{v}_p^{-1} \left\{k^2 \left[c_1 q_S \bar{q} + q_\Psi (v_{c,S} - v_{o,S})\right] + ikq_S v_{c,N}\right\} \tag{20b}$$

The only term that differs intrinsically from (10) is a_2, which now has an imaginary part. We see that the real part of a_2 is still invariable positive, so the real part of $a_1 - 4a_2$ is smaller than a_1. However, we can no longer conclude that the real part of λ has the same sign as a_1 ~~except in appropriate limits.~~

~~For short wavelengths (large k), the system is stable with $\Re(\lambda) < 0$: the two limiting forms of the two eigenvalues are~~

$$\cancel{\lambda_1 \sim -\bar{v}_p q_\Psi k^2, \qquad \lambda_2 \sim -\frac{c_1 q_S q + q_\Psi (v_{c,S} - v_{o,S})}{q_\psi},}$$

~~and the assumptions about q_ψ, q_S, $v_{o,S}$ and $v_{c,S}$ in –ensure that both of these expressions are negative. Limited storage over short length scales prevents large variations in water discharge and prevents short-wavelength disturbances in conduit size and effective pressure from growing.~~

~~Conversely, at sufficiently large wavelengths (small k), we have $a_1 \sim (c_1 q_S + v_{o,S} - v_{c,S})$, $a_2 \sim O(k)$, and a channel-like conduit will be unstable with one eigenvalue behaving as $\lambda \sim (c_1 q_S + v_{o,S} - v_{c,S})$, while a cavity-like conduit is stable at long wavelengths. This is simply~~ We can identify two mechanisms for instability. The first is essentially the same as Nye's ~~instability at work. Clearly, then, there is a critical wavelength at which a channel-like conduit with distributed water storage will become unstable (since it is stable at short wavelengths and unstable at long wavelengths), and the critical wavelength is determined by the various parameters in the model. How large that wavelength is matters, because we have assumed in setting up our stability analysis that we are looking at only a relatively small part of the domain, with periodic boundary conditions. If the critical wavelength approaches the full domain size L, our simplified analysis breaks down, and the instability is no longer guaranteed to occur. This is consistent with figure 7 of Schoof et al. (2014), where channel-like conduits eventually become stable for large enough water supply rates.~~

~~Notably, the critical wavelength for instability of channel-like conduits will depend on storage capacity $\bar{v}_p$. The~~ melt-drainage feedback: the more storage capacity there is, the smaller all the terms containing k are in (20) for a given wavenumber k, and the more likely the first term $(c_1 q_S + v_{o,S} - v_{c,S})$ in the definition of a_1 is to dominate the eigenvalue λ, leading to instability for a channel-like system. To increase the size of the potentially stabilizing terms therefore requires larger wavenumbers

k, and ~~therefore~~ a larger range of short wavelengths is likely to be unstable. Although we are not able to address a system of finite length directly with ~~this approach , our results indicate~~ our approach of a quasi-periodic domain, the results of Schoof et al. (2014) confirm that systems of limited size ~~may~~ can remain stable if storage capacity is sufficiently limited ~~, and this is confirmed by~~ Schoof et al. (2014).

~~However, an interesting possibility is left open by . Even~~ A second instability mechanism can occur if the conduit is cavity-like with $(c_1 q_S + v_{o,S} - v_{c,S}) < 0$ and therefore $a_1 < 0$~~, it may still be~~ . It is still possible to have an eigenvalue with positive real part and hence instability~~. This is the case~~ , because a_2 has a non-zero imaginary part.

One particular case in which an instability due to this term may occur is with limited storage capacity (so $\bar{v}_p^{-1}$ is large) and at an intermediate wavelengths range (so k is small but not too small). ~~To be definite in identifying that parameter regime, suppose we scale the model; with the constitutive relations , this is done by defining scales~~ ~~$[S]$, $[N]$, $[q]$, $[x]$ and $[t]$ through $c_3[S]^\alpha[\Psi]^{1/2} = [q]$, $[S]/[t] = c_1 c_3 [S]^\alpha [\Psi]^{3/2} = c_2[S][N]^n$, $[\Psi] = [N]/[x] = \Psi_0$ and a constraint on $v([N])$ through the prescribed quantity of water in the system. Dimensionless variables are then $N^* = [N]^{-1}N$, $S^* = [S]^{-1}S$, $q^* = [q]^{-1}q$, $x^* = [x]^{-1}x$, $t^* = [t]^{-1}t$. Dropping asterisks, the model then becomes with but with periodic boundary conditions, and with $c_1 = c_2 = c_3 = 1$ and $\Psi_0 = 1$. The stability analysis applies as stated, and the derivatives q_S, q_Ψ, $v_{c,S}$, $v_{c,N}$ and $v_{o,S}$ are $O(1)$ quantities.~~ Then it is possible, purely by controlling the size of the storage capacity $\bar{v}_p$ ~~is replaced by a dimensionless counterpart~~

$$\tilde{v}_p = v_p c_1^{(n+2)/n} c_2^{-2/n} c_3^{(2-n)/\alpha} \bar{q}^{\prime 2(\alpha-1)-n]/\alpha|}$$

$$\times \Psi_0^{(2\alpha m + 4\alpha + 2 - n)/(2\alpha n)},$$

~~and k as well as λ are scaled wavenumbers and growth rates.~~

~~We can make a cavity-like conduit unstable provided $k \ll 1$ and $\tilde{v}_p \ll k^3$. Then $a_2 \sim i\tilde{v}_p^{-1} k q_S v_{c,N}$ and $a_1^2 - 4a_2 \sim [(q_S + v_{o,S} - v_{c,S}) -$~~ ~~From our constraints on $\tilde{v}_p$ and k, $k \ll 1$ and $\tilde{v}_p^{-1} k \gg \tilde{v}_p^{-2} k^4$ so~~

$$a_1^2 - 4a_2 \sim -4i\tilde{v}_p^{-1} k q_S v_{c,N}.$$

~~Then we have~~

$$\lambda = \frac{1}{2}\left(a_1 \pm \sqrt{a_1^2 - 4a_2}\right)$$

$$\simeq \frac{1}{2}\left[(q_S + v_{o,S} - v_{c,S}) - \tilde{v}_p^{-1} k^2 q_\Psi \pm 2i \frac{1+i}{\sqrt{2}} \sqrt{\tilde{v}_p^{-1} k q_S v_{c,N}}\right]$$

~~But with our assumptions on k and $\tilde{v}_p$, this is~~

$$\lambda \sim \pm(1-i)\sqrt{\frac{\tilde{v}_p^{-1} k q_S v_{c,N}}{2}}$$

, to ensure that $a_1^2 - 4a_2 \sim -i4\bar{v}_p^{-1} q_S v_{c,N} k$ and

$$\lambda \sim \pm(1-i)\sqrt{\frac{\tilde{v}_p^{-1} k q_S v_{c,N}}{2}}. \tag{21}$$

Full details can be found in section 6 of the supplementary material. Choosing the $+$ sign ensures an eigenvalue with positive real part. Importantly, this instability corresponds to a growing wave that propagates, in this case downstream as the imaginary part of λ is then negative. It is also of the same size as the real part, so propagation is not slow. This unstable wave is not the result of Nye's instability as the conduit is cavity-like. Instead, it is the result of an interaction between the dependence of the conduit closing rate on effective pressure and the dependence of water drainage (which affects effective pressure through water storage) on conduit size~~: the dominant balance that underpins it is~~

$$\cancel{\frac{\partial S'}{\partial t} \sim v_{c,N} N',}$$

$$\cancel{-\bar{v}_p \frac{\partial N'}{\partial t} + q_S \frac{\partial S'}{\partial x} \sim 0,}$$

~~where the perturbed conduit evolution does not include the melting term $c_1 q \Psi$ at all. Equations can be combined into the single 'diffusion' equation $\bar{v}_p \partial^2 N'/\partial t^2 - q_s v_{c,N}^{-1} \partial N'/\partial x \sim 0$, except that the roles of time t and space x have been reversed. We can identify the positive feedback causing growth as a the result of a phase lag, where S' leads N' in phase and thus ensures that $\partial S'/\partial x$ is positive where N' has a maximum, therefore ensuring that $\partial N'/\partial t$ is also positive.~~

~~Naturally, this sketch is incomplete, as the 'diffusion' problem with the roles of x and t reversed is not well-posed. The growth rate in grows unboundedly as wavelength k^{-1} approaches zero, which is a clear sign something is amiss. The discussion above was merely designed to identify a positive feedback that can drive growth; a negative feedback is necessary to ensure there is a fastest growing wavelength and that short wavelengths are damped as already discussed~~.

6 Conclusions

As demonstrated previously using a much more restricted sweep of parameter space in Schoof et al. (2014), we have shown that drainage systems capable of switching spontaneously between channel- and cavity-like behaviour are stable in the presence of a localized water reservoir at low and high water throughput, with an unstable intermediate range of water fluxes. In that range, spontaneous oscillations in reservoir level will occur, driven by Nye's (1976) instability mechanism driving outburst floods. These outburst floods turn out to be regular, periodic oscillations in water level at least at low-to-moderate water input, where a simplified, 'lumped' model of reservoir drainage generally reproduces the results of a more sophisticated, spatially-extended drainage model. At high water throughput rates, our results are more equivocal as to the emergence of such limit cycles in the model used; in any case, the model necessarily breaks down physically (as opposed to mathematically) because it predicts that the oscillations will invariably reach negative effective pressures and therefore flotation of the ice dam at such large throughput rates.

It is worth pointing out that part of our focus has been on the emergence of limit cycle solutions in order to identify how the flood initiation mechanism can prevent flood magnitude from increasing progressively from cycle to cycle as observed in Ng (1998) and Fowler (1999), and to present an alternative mechanism for suppressing the continued growth in amplitude from that considered by Fowler (1999), whose lake is necessarily 'sealed' between floods. ~~That mechanism~~ The alternative mechanism here is the ability of a cavity-like drainage system that remains during the reservoir recharge phase to switch to a channlized drainage mode.

The fact that we illustrate this by showing evolution towards a limit cycle is not at odds with the chaotic behaviour observed by Kingslake (2015) using a variant of Fowler's (1999) model: the chaotic behaviour is intrinsically the result of a time-varying water input to the reservoir, which we have not studied in this paper. It is entirely possible, and in fact likely, that our model will also behave chaotically under such time-varying water input rates $q_{in}(t)$. The point of demonstating that limit cycles emerge was rather to underline that the growth of Nye's (1976) instability is bounded in our model without having to appeal to additional physics.

The lower cut-off to the drainage instability that leads to outburst floods corresponds to the drainage system switching to a cavity-like state under steady flow conditions when water input to the reservoir can be drained steadily by those cavities. The mechanism for the cut-off at high water throughput rates is harder to identify. The lumped model predicts that the high sensitivity of water level in the reservoir to the evolution of the draining conduit will induce water pressure gradients that reduce flow as the conduit grows, and therefore suppress its enlargement due to heat dissipation. The threshold at which this happens is however significantly underestimated by lumped model relatively to its spatially-extended counterpart, which develops more wave-like instabilities at higher water throughput rates.

In closing, we have also investigated how wave-like instabilities can occur when the water reservoir is not localized but spread out or 'distributed' along the flow path (for instance, in the form of many small reservoirs like basal crevasses). This type of instability was first observed in Schoof et al. (2014), and persists even where water throughput is insufficient to lead to channel-like behaviour. Adapting the stability analysis performed on a model with localized storage, we have shown that Nye's instability persists, but also that a second instability mechanism emerges, in which a phase shift between water storage and flux arises that causes water to accumulate in regions where effective pressure is already low.

Future work is likely to focus on capturing the role of overpressurization of the drainage system in initiating and mediating the instability driving outburst floods, since flood initiation at water pressures below flotation is confined to a relatively small part of parameter space, and the model predicts that reaching zero effective pressure and initiation by partial flotation of the ice dam is likely to be common.

7 Code availability

The MATLAB code used in the computations reported is included in the supplementary material, except for the code used to solve the transient calculations displayed in figures 9–11, which is included in the supplementary material to Rada and Schoof (2018)
.

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Appendix A: Asymptotic solutions for large lakes with moderate inflow

This appendix provides only a brief sketch of the derivation of an asymptotic solution for a limit cycle solution in the case where the reservoir volume is large and inflow is sufficient to ensure that the reservoir cannot be drained by a cavity-like conduit, but also that the conduit size evolves much faster than the time scale over which the reservoir fills. Full details are given in section 4 of the supplementary material.

The solution is developed from a parameteric limit of the model (2) with (1g), where we assume $S_0 = \infty$. We scale as $N^{**} = N/[N]'$, $S^{**} = S/[S]'$, $t^{**} = t/[t]$, where the scales are defined through $[S]'/[t]' = c_1 c_3 [S]'^{\alpha} \Psi_0^{3/2} = c_2 [S]'[N]'^{n}$ and $V_p [N]'/[t]' = c_1 [S]'^{\alpha} \Psi_0^{1/2}$. Substituting and omitting the asterisks immediately, we obtain

$$\dot{S} = S^{\alpha}|1-\nu N|^{3/2} + \delta - S|N|^{n-1}N \tag{A1a}$$

$$\dot{N} = -\epsilon + S^{\alpha}|1-\nu N|^{-1/2}(1-\nu N) \tag{A1b}$$

where $\nu = [N]'/(\Psi_0 L)$, $\delta = u_b h_r/(c_2[S]'[N]'^{n})$ and $\epsilon = q_{in}/(c_1[S]'^{\alpha}\Psi_0^{1/2})$.

We assume that the exponents n and α satisfy $n+1 > \alpha > 1$, in which case the asymptotic solution we develop is valid when

$$\delta \ll \epsilon \lesssim \delta^{(\alpha-1)(n+1)/(\alpha n)} \ll 1$$

and $\nu \lesssim 1$.

The main flood phase is described by omitting terms of $O(\delta)$ and $O(\epsilon)$

$$\dot{S} = S^{\alpha}|1-\nu N|^{3/2} - S|N|^{n-1}N, \tag{A2a}$$

$$\dot{N} = S^{\alpha}|1-\nu N|^{-1/2}(1-\nu N), \tag{A2b}$$

where matching with the initiation phase described later requires that $(N,S) \to (0,0)$ as $t \to -\infty$. The transformation $P = N/S$ allows the system (A2) to be re-cast in such a way as to demonstrate that the orbit into the fixed point $(0,0)$ is unique, so there is only one flood phase solution. The orbit terminates at a finite $N = \tilde{N}_f$ as $t \to \infty$; this then sets the amplitude of the lake level fluctuations that give the asymptotic formula for the flood cycle period (14).

The refilling phase is described by the rescaling $\tilde{N} = N$, $\tilde{S} = \delta^{-1} S$ and $\tilde{t} = \epsilon(t - t_f)$, where t_f is the time of the last flood. At leading order,

$$0 = 1 - \tilde{S}|\tilde{N}|^{n-1}\tilde{N}, \tag{A3a}$$

$$\frac{\mathrm{d}\tilde{N}}{\mathrm{d}\tilde{t}} = -1; \tag{A3b}$$

conduit size is quasi-steady and cavity-like, while lake level evolves purely because of inflow.

The refilling phase ends as $N \to 0$ and cavity size becomes large. The relevant rescaling becomes

$$\hat{N} = \delta^{-(\alpha-1)/(\alpha n)}\tilde{N}, \qquad \hat{S} = \delta^{(\alpha-1)/\alpha}\tilde{S},$$

$$\hat{t} = \delta^{-(\alpha-1)/(\alpha n)}(\tilde{t} - N_f)$$

and gives at leading order

$$\epsilon\delta^{-(\alpha-1)(n+1)/(\alpha n)} \frac{\mathrm{d}\hat{S}}{\mathrm{d}\hat{t}} = \hat{S}^\alpha + 1 - \hat{S}|\hat{N}|^{n-1}\hat{N}, \tag{A4a}$$

$$\frac{\mathrm{d}\hat{N}}{\mathrm{d}\hat{t}} = -1, \tag{A4b}$$

where we have assumed $\epsilon \sim \delta^{(\alpha-1)(n+1)/(\alpha n)}$; the alternative case of $\epsilon \sim \delta$ is described in the supplementary material. The key to (A4) is that it predicts finite-time blow-up in $\hat{S}$ at some finite $\hat{N} = \hat{N}_c$ that depends purely on the value of $\epsilon\delta^{-(\alpha-1)(n+1)/(\alpha n)}$. This is the smallest value of effective pressure (rescaled, of course) that is reached during the drainage cycle. The larger $\epsilon \sim \delta^{(\alpha-1)(n+1)/(\alpha n)}$, the smaller and eventually more negative $\hat{N}_c$ becomes; there is therefore a critical value of $\epsilon \sim \delta^{(\alpha-1)(n+1)/(\alpha n)}$ at which $\hat{N}_c = 0$. When rendered in dimensional form, this value gives the formula (15).