Reply to "Basal buoyancy and fast-moving glaciers: in defense of analytic force balance" by C. J. van der Veen (2016)

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Abstract. Two approaches to ice-sheet modeling are available. Analytical modeling is the traditional approach (Van der Veen, 2016). It solves the force (momentum), mass, and energy balances to obtain three-dimensional solutions over time, beginning with the Navier-Stokes equations for the force balance. Geometrical modeling employs simple geometry to solve the force and mass balance in one dimension along ice flow (Hughes, 2012a). It is useful primarily to provide the first-order physical basis of ice-sheet modeling for students with little background in mathematics. The geometric approach uses changes in ice-bed coupling along flow to calculate changes in ice elevation and thickness, using floating fraction ϕ along a flowline or flowband, where $\phi = 0$ for sheet flow, $0 < \phi < 1$ for stream flow, and $\phi = 1$ for shelf flow. An attempt is made to reconcile the two approaches.

Introduction

Cornelis "Kees" Van der Veen's comparison of geometric and analytic approaches to the force balance in glaciology in *The Cryosphere* (Van der Veen, 2016) is most welcome because he takes seriously my geometrical approach to the longitudinal force balance, citing many of my paper from when I first introduced the concept (Hughes, 1992) to the latest application (Hughes et al., 2016). To begin, the analytic force balance is not challenged by me. The geometric force balance is useful only for one-dimensional flow along ice-sheet flowlines or flowbands of constant width. For two-dimensional flow in the map plane, width become a variable and geometrical areas become geometrical volumes; substantially increasing geometrical complexity with little advance in physical insight. The analytic force balance is typically obtained by solving the Navier-Stokes equations, which can be done in three dimensions and, when including the mass and energy balances, becomes time-dependent. The geometrical approach is useful for understanding the force balance by comparing the areas of right triangles and rectangles (or parallelograms).

Addressing Van der Veen (2016)

My interest in the force balance for ice sheets spans four decades, beginning when I used glacial geology to reconstruct former ice sheets from the bottom up based on the strength of ice-bed coupling deduced from glacial geology, an approach that also produced the concave surface of ice streams for the first time (Denton and Hughes, 1981, Chapters 5 and 6). I developed the geometric approach after observing the huge arcing transverse crevasses at the head of Byrd Glacier, and realized it was actually pulling ice out of the East Antarctic Ice Sheet (Hughes, 1992). Since then it has been a work in progress. Van der Veen (2016) cites earlier stages of that work (Hughes, 2003, 2008). I would prefer that he use my current treatment in Hughes (2012a) and Hughes et al. (2016).

Referring to Hughes (2008), Van der Veen (2016) states on his page 1332 that I believe lateral drag vanishes at the center of an ice stream. Lateral shear stress σ_{xy} vanishes, but the lateral shear force does not. On one side, stress σ_{xy} acts on side area A_y and on the other side stress $-\sigma_{xy}$ acts on side area $-A_y$, with A_y and $-A_y$ being vectors in opposite y directions, so the shear force is always positive and opposes longitudinal gravitational forcing.

Van der Veen (2016) states his Eq. (9) is my Eq. (36) in Hughes (2003). It is not, his signs are different from mine and his σ_F is not the same as my σ_T. In the geometric force balance, the driving force is the area of a triangle and all the resisting forces are areas of triangles and a rectangle (or parallelogram) that fit into the triangle so the driving and resisting forces are identical. All signs are positive in my Eq. (36). His σ_F is my flotation stress, which doesn't appear in my 2003 paper. It appears in my Nova book, *Holistic Ice Sheet Modeling* (Hughes, 2012a) and in Hughes et al. (2016) in *The Cryosphere*. Van der Veen (page 1333) states my σ_F is his $\tilde{R}_{xx}$. It is not. His force budget approach has no way for calculating my flotation stress σ_F because his approach has no place for my floating fraction ϕ of ice under an ice stream (which he calls a "basal buoyancy factor" that obscures its physical meaning), see my Fig. 1.

Van der Veen (2016) states his Eqs. (13), (14), and (15) are my equations in my 2008, 2012a, and 2016 publications. They are not. His signs are different from mine and even some of his terms are different from mine. The proof is found by substituting his Eqs. (13) through (15) into his Eq. (9), which does not deliver 0 = 0 for the force balance. My equations, reproduced as my Table 1 from Table 12.1 in Hughes (2012a), do give 0 = 0. In my geometric force balance, resisting forces are represented by triangles and a rectangle (or parallelogram) that exactly fit inside a big right triangle that represents my driving force, so the area of my big triangle is the same as summed component areas from resisting forces within it. Therefore 0 = 0 *must* be obtained, see my Fig. 2.

Van der Veen (2016) plots his Eqs. (9) through (15) in his Fig. 2, so they cannot represent my force balance because they are not my equations. Also the plot of his "Gradients in longitudinal stress" should be gradients in longitudinal force, which is a stress, so he can compare stresses with stresses, not with stress gradients of stresses. If his Fig. 2 truly plots a longitudinal stress gradient, it compares apples with oranges. Also in his Fig. 2, his longitudinal stress (or force) gradient acts in the same direction as his gravitational driving force. That is impossible in my geometric force balance, see my Fig. 2.

Referring to my Figure 3 (left), Figure 3 in Van der Veen (2016), line AF should be parallel to line BE because they both show ice pressure increasing linearly with depth. Line CE shows how water pressure increases linearly with depth, as is obvious at the calving front. In my geometrical force balance, the longitudinal gravitational driving force is area ADF of the big triangle. Fitted inside ADF are a resisting flotation force given by area BDE for floating ice fraction ϕ and a resisting drag force given by area ABEF for the grounded ice fraction $1-\phi$ in my Fig. 1. Inside BDE is area CDE for the resisting force from water pressure and area BCE for the resisting force from the tensile strength of ice. Inside area ABEF is the triangle above B for basal drag and the parallelogram below B for side drag. Resistance from basal drag is the area of the triangle above B. Resistance from side drag is the area of the parallelogram below B if lines BE and AF are made parallel. If BE is made part of AF a rectangle would replace the parallelogram but the area would be unchanged, see my Fig. 2. That's all there is to it. The only remaining task is to replace forces with products of stresses and lengths (for areas having unit or constant widths along *x*) upon which the stresses act along a flowline (no width) or a flowband (constant width). My solution for the force balance is exact because forcing area ADF equals resisting areas ABEF, BCE, and CDE inside ADF. All gravitational and resisting forces in the longitudinal direction of ice flow are thereby included, with ABEF representing the force from both basal and side drag.

Van der Veen (2016) correctly states his Eq. (16) represents my longitudinal gravitational driving force, but then he states it "does not represent the gravitational driving force" (page 1335). It does. In my direction $-x$ of ice flow, the gravitational force (a horizontal vector) is the average ice pressure (a scalar) times the transverse cross-sectional area against which it acts (as a horizontal vector in my $-x$ direction), which for an ice stream of constant width is ice width times ice height above the bed, a height that varies along *x*, as does average ice pressure, so the gravitational driving force varies along *x*. The correct representation of my longitudinal geometric force balance is my Fig. 2 where his area ABEF is my area 1+2 for basal and side drag at *x*.

Van der Veen (2016) states on his page 1335 that a longitudinal force balance along *x* must be made over incremental distance Δx that shrinks to zero. My longitudinal force balance along *x does* in my Fig. 2 (bottom), see Hughes (2012a, Appendix G) and Hughes et al. (2016, page 10). I subtract longitudinal force areas over distance Δx to get my longitudinal force balance Eq. (22) in Hughes et al. (2016). However, Van der Veen (2016) is incorrect in stating a longitudinal force balance *always* must be made over length Δx. At the calving front of an ice shelf the balance is obtained right at the calving front where $\Delta x = 0$, as Robin (1958) proved 59 years ago *geometrically*.

Van der Veen (2016) discusses areas ADF and APD in terms of "lithostatic stresses" increasing with depth in his Fig. 4(a), shown in my Fig. 3 (right). The areas are forces. As he shows by his horizontal arrows in his Fig. 4(a), area ADF is my horizontal gravitational driving force and area APD is the sum of my horizontal resisting forces opposing the driving force in my geometrical force balance shown in my Fig. 2 (center) with an ice surface slope at *x*. His area APD can be subdivided into my smaller areas of triangles and a rectangle in my Fig. 2 (center) to obtain areas that resist gravitational forcing from his area ADF. There is no surface slope in his Fig. 4(a), a condition that applies to an unconfined linear ice shelf having constant thickness (Weertman, 1957; Robin, 1958), in which case only my areas 3 and 4 in my Fig. 2 (bottom) add to give his area APD since there are no basal and side drag forces represented by my areas 1 and 2. Raymond (1982) analyzed deformation near interior ice divides where the surface slope is also zero.

Van der Veen (2016) correctly shows the geometrical force balance in my Fig. 2 (bottom) for a sloping ice surface above a horizontal bed in his Fig. 4(b), shown in my Fig. 3

(right). From these figures we can both obtain the geometric longitudinal force balance over incremental length Δx in analytic form when $\Delta x \rightarrow 0$. In my Fig. 2 (bottom), my big triangles at x and $x + \Delta x$ are gravitational driving forces that are respectively subdivided into areas 1, 2, 3, 4 and areas 5, 6, 7, 8 that resist gravitational motion along *x*.

My Geometrical Force Balance

I developed the geometrical force balance to teach the fundamentals of glaciology to students with an inadequate background in mathematics, usually students studying to be glacial geologists (Hughes, 2012a). My geometrical approach was designed to make maximum use of glacial geology in reconstructing former ice sheets from the bottom up (Hughes, 1998, Chapters 9 and 10; Fastook and Hughes, 2013) and in demonstrating how basal thermal conditions produce glacial geology under the Antarctic Ice Sheet today (Hughes, 1998, Chapter 3, Wilch and Hughes, 2000; Siegert, 2000). Previously I had spent more time teaching calculus than glaciology because the Navier-Stokes equations had to be integrated in the force balance.

The major variable in my geometrical force balance is the floating fraction ϕ of ice, where $\phi = 0$ for sheet flow, $0 < \phi < 1$ for stream flow, and $\phi = 1$ for shelf flow. Here we are primarily interested in stream flow as shown in my Fig. 1 for possible ϕ distributions at the bed and my Fig. 2 for the longitudinal force balance. From Newton's second law of motion in a vertical force balance, gravitational force F_G at the base must be the same for floating area $w_F \Delta x$ and total area $w_I \Delta x$ such that $F_G = (\rho_I h_I w_F \Delta x) g = (\rho_I h_F w_I \Delta x) g$ for ice density ρ_I and gravity acceleration g to obtain basal pressures $P_F = \rho_I g h_F$ and $P_I = \rho_I g h_I$ that support ice of respective floating and total heights h_F and h_I. This vertical force balance is satisfied if h_F goes from 0 to h_I as w_F goes from 0 to w_I. The basal water pressure is $P_W = \rho_W g h_W = P_F = \rho_I g h_F$ for water density ρ_W and water height h_W needed to float ice height h_F. The floating fraction of ice at *x* is therefore:

$$\phi = w_F / w_I = h_F / h_I = P_F / P_I = P_W / P_I .$$

Pulling force $\sigma_T h_I$ resists the gravitational driving force given by area 4 in Figure 2 (bottom), which is area 3+4 minus area 3. Area 3+4 is one-half flotation height $h_F = h_I \phi$ times basal floating length $P_F = P_I \phi$, so area 3+4 is $\bar{P}_I h_I \phi^2$. Area 3 is one-half height $h_W = (\rho_I / \rho_W) h_F = (\rho_I / \rho_W) h_I \phi$ times the same basal floating length $P_F = P_I \phi$. Then the tensile pulling stress is $\sigma_T = \bar{P}(1 - \rho_I / \rho_W)\phi^2$. It is that simple. At the calving front where $\phi = 1$ this is the solution obtained by Weertman (1957) and Robin (1958). Table 1 lists all stresses resisting gravitational forcing at *x*.

At distance *x* from the ice-shelf grounding line in my Fig. 2, gravitational driving force $F_G = \bar{P}_I h_I$ is resisted by the sum of upstream tensile pulling force $F_T = \sigma_T h_I$ and downstream compressive pushing force $F_C = \sigma_C h_I$ so $\sigma_T = \bar{P}_I - \sigma_C$. Tensile force $\sigma_T h_I$ balances the part of the driving force equal to area 4, and resisting force $\sigma_C h_I$ balances the part of the driving force equal to areas 1+2+3 in Figure 2 (center and bottom), and includes all downstream resistance due to averaged basal and side shear stresses $\bar{\tau}_O$ and $\bar{\tau}_S$ respectively linked to areas 1 and 2, plus local water buttressing stress σ_W linked to area 3, all of which resist gravitational forcing equivalent to these areas.

My geometrical force balance is shown in Fig. 2, which is Fig. 5 in Hughes et al. (2016). Along incremental length Δx, change ΔF_G in the longitudinal gravitational driving force F_G is balanced by change ΔF_T in the tensile pulling force F_T plus change ΔF_W in the water buttressing force F_W plus basal drag force F_O plus side drag force F_S, where $F_F = F_T + F_W$ is a flotation force that requires ice-bed uncoupling by basal water. Dividing by Δx and letting $\Delta x \to 0$ gives as the longitudinal gravitational force gradient

$$\partial F_G / \partial x = \partial(\bar{P}_I h_I) / \partial x = P_I \alpha_I = \partial(\sigma_F h_I) / \partial x + \tau_O + 2\tau_S (h_I / w_I)$$

where the bed is represented by an up-down staircase with successive Δx steps so ice thickness gradient α_I equals α for ice surface slope on each step, P_I is the overburden ice pressure at the base, τ_O is the basal shear stress, τ_S is the side shear stress for two sides, h_I is ice thickness, h_W is the height of water that floats flotation height h_F of ice supported by basal water pressure P_W such that $P_W = P_F$ and $h_W = (\rho_I / \rho_W) h_F$ for floating fraction ϕ, and my flotation stress $\sigma_F = \sigma_T + \sigma_W = \bar{P}_I \phi^2$ for ice tensile stress σ_T and water buttressing stress σ_W, all at distance *x* upstream from an ice-shelf grounding line. At the calving front of an ice shelf where $\phi = 1$ so $h_F = h_I$ this is identical to the Weertman (1957) and Robin (1958) solutions. Together σ_T and σ_W resist gravitational forcing linked to $\bar{P}_I$ in an ice shelf and $\bar{P}_I \phi^2$ linked to floating fraction ϕ in an ice stream at *x*. My σ_F differs from R_{xx} in Equation (1) of Van der Veen (2016) because my σ_F always requires basal water deep enough to uncouple ice from the bed or to supersaturate basal till. In ice streams, water height h_W above the bed is the height to which basal water would rise in a borehole, including heights far above sea level (Kamb, 2001).

Resistance from my σ_W may be akin to bridging stresses across water-filled cavities discussed by Van der Veen (2016). The existence of σ_W in the geometric force balance is not readily apparent from analytic solutions of the Navier-Stokes equations, but Van der Veen (2016) may have teased it out with his bridging stress, which forces him to add resistance by including steep shear-stress gradients on each side of his cavities. He maintains his cavities are small so these gradients average out to zero along an ice stream, eliminating the need for my σ_W. They cannot average to zero if his cavities are water-filled and get bigger and closer together downstream, as required to progressively uncouple ice from the bed. Then cavities themselves have a size and distribution gradient. Figure 1, which is Figure 4 in Hughes et al. (2016), shows my concept of water-filled cavities in area $w_I \Delta x$ under an ice stream. We do not know which concept of cavities is correct.

Concluding Remarks

I developed the geometrical force balance over some decades, from Hughes (1992) through Hughes et al. (2016). My papers are a work in progress, see pages 201-202 of Hughes et al. (2016) regarding h_W, h_F, σ_W, and σ_F not included in earlier papers. To access my most recent thinking, see Hughes (2012a) and Hughes et al. (2016). All the earlier studies are flawed in various ways. The last ones may also have flaws I haven't detected. Some criticisms by Van der Veen (2016) are directed at my earlier flawed papers.

This response gives me an opportunity to correct three mistakes in Hughes (2012a) that will be apparent to careful readers. The first line in Equation (12.9) should be:

$$\partial(\sigma_F h_I)/\partial x = \partial\left[\frac{1}{2}\rho_I g h_I^2 \phi^2\right]/\partial x = P_I \phi(\phi\alpha_I + h_I \partial\phi/\partial x)$$

and in the second line ϕ should be ϕ^2. In the denominator of Equation (17.18), r should be replaced by $(a-r)$. The first line of Equation (22.18) should be:

$$\Delta h_i^* / \Delta x = \phi^2 \left(\frac{\Delta h_I}{\Delta x}\right)_i + \left(\frac{h_I}{2}\right)_i \frac{\Delta\phi^2}{\Delta x} + \frac{(\tau_O)_i}{\rho_I g h_I^*} + \frac{2(\tau_S)_i}{\rho_I g w_I} = \frac{(\tau_O^*)_i}{\rho_I g h_I^*}$$

Equation (22.18) applies to sheet flow when $\phi = \partial\phi/\partial x = 0$ and τ_O^* increases resistance from basal drag τ_O by including side drag τ_S in flowbands having some side shear. If $\phi > 0$ in tributaries supplying ice streams, and since tributaries are ubiquitous in the sheet-flow interior of the Antarctic Ice Sheet (Hughes, 2012b), side shear must be taken into account even for sheet flow because tributaries are flowbands.

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Table 1: *Resisting Stresses Linked to Floating Fraction* $\phi = P_F/P_I$ *of Ice and Gravitational Forces Numbered in Figure 2 for the Geometrical Force Balance.*

Basal water pressure at *x*, from gravity force 3: $P_W = \rho_W g h_W$
Ice overburden pressure at *x*, from gravity force (1+2+3+4): $P_I = \rho_I g h_I$
Upslope tensile stress at *x*, from gravity force 4: $\sigma_T = \bar{P}_I \left(1 - \rho_I / \rho_W\right)\phi^2$
Downslope compressive stress at *x* due to $\bar{\tau}_O$ and $\bar{\tau}_S$ along *x* and σ_W at *x* = 0: $\sigma_C = \bar{P}_I - \sigma_T = \bar{P}_I - \bar{P}_I \left(1 - \rho_I / \rho_W\right)\phi^2$
Downslope water-pressure stress at *x*, from gravity force 3: $\sigma_W = \bar{P}_I \left(\rho_I / \rho_W\right)\phi^2$
Upslope flotation stress at *x* from gravity force (3+4): $\sigma_F = \sigma_T + \sigma_W = \bar{P}_I \phi^2$
Longitudinal force balance at *x* from gravity force [(5+6+7+8)–(1+2+3+4)]: $P_I \alpha = \partial\left(\sigma_F h_I\right)/\partial x + \tau_O + 2\tau_S\left(h_I / w_I\right)$
Flotation force gradient at *x* from gravity force [(7+8)–(3+4)]: $\partial\left(\sigma_F h_I\right)/\partial x = P_I \phi\left(\phi \alpha_I + h_I \partial\phi / \partial x\right)$
Basal shear stress at *x* from gravity force (5–1): $\tau_O = P_I \left(1 - \phi\right)^2 \alpha - P_I h_I \left(1 - \phi\right)\partial\phi / \partial x$
Side shear stress at *x* from gravity force (6–2): $\tau_S = P_I \left(w_I / h_I\right)\phi\left(1 - \phi\right)\alpha + \bar{P}_I w_I \left(1 - 2\phi\right)\partial\phi / \partial x$
Average downslope basal shear stress to *x* from gravity force 1: $\bar{\tau}_O = \bar{P}_I w_I h_I \left(1 - \phi\right)^2 / \left(w_I x + A_R\right)$
Average downslope side shear stress to *x* from gravity force 2: $\bar{\tau}_S = P_I w_I h_I \phi\left(1 - \phi\right)/\left(2\bar{h}_I x + 2L_S \bar{h}_S + C_R \bar{h}_R\right)$

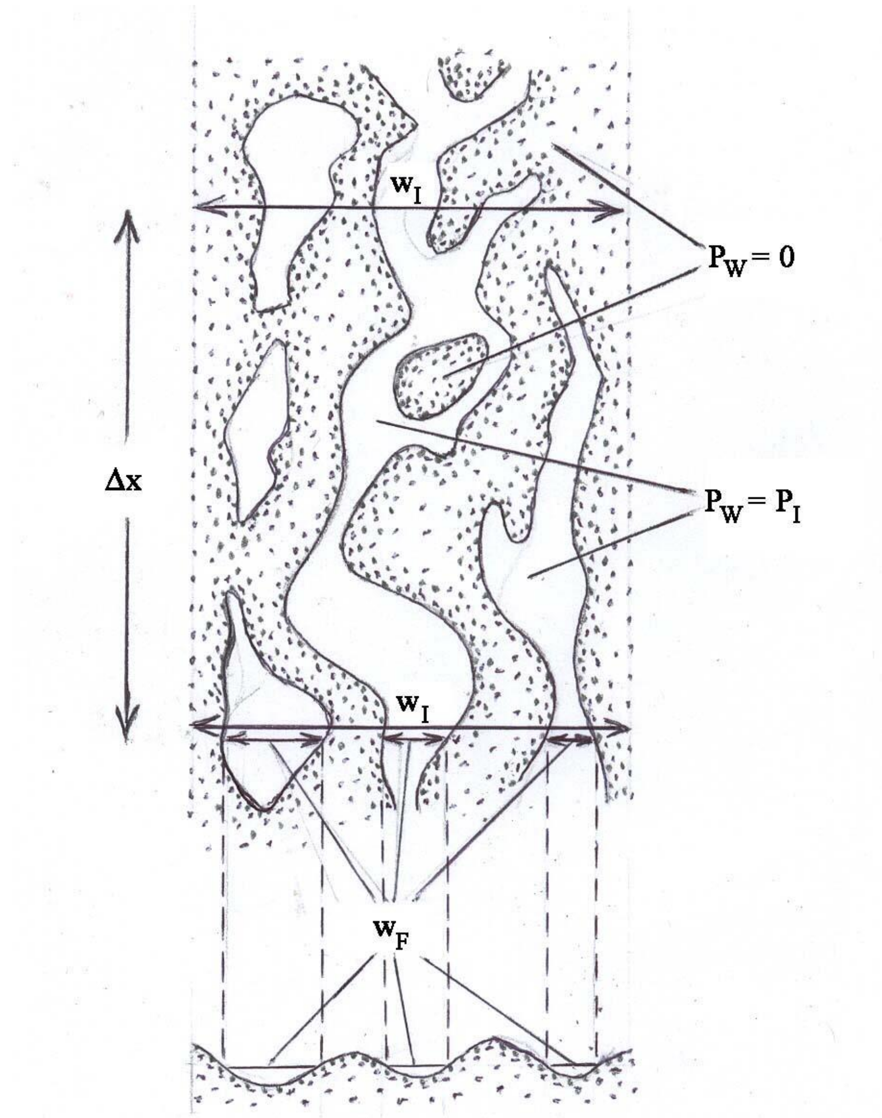

Figure 1: Figure 4 from Hughes et al. (2016). Under an ice stream, basal ice is grounded in the shaded areas and floating in the unshaded areas (top) as seen in a transverse cross-section (bottom) for incremental basal area $w_I \Delta x$.

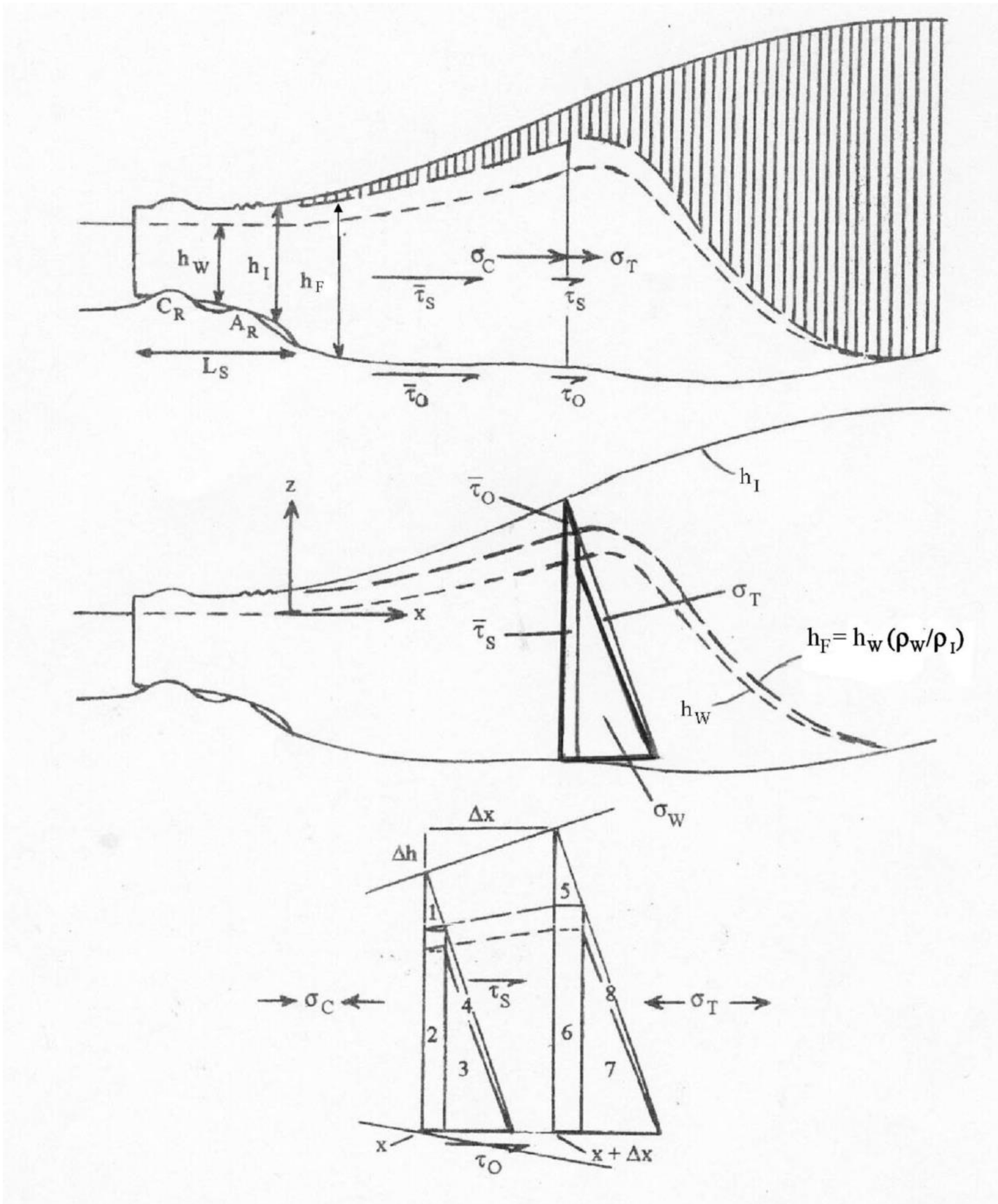

Figure 2: Figure 5 from Hughes et al. (2016). Top: Stresses at *x* and downstream from *x* that resist gravitational forcing. The bed supports ice in the shaded area. Middle: The gravitational force inside the thick border is linked to σ_C which represents all downstream resistance to ice flow at point *x*. Bottom: Gravitational forces (geometrical areas 1 through 8) and resisting stresses along incremental downstream length Δx at point *x*.

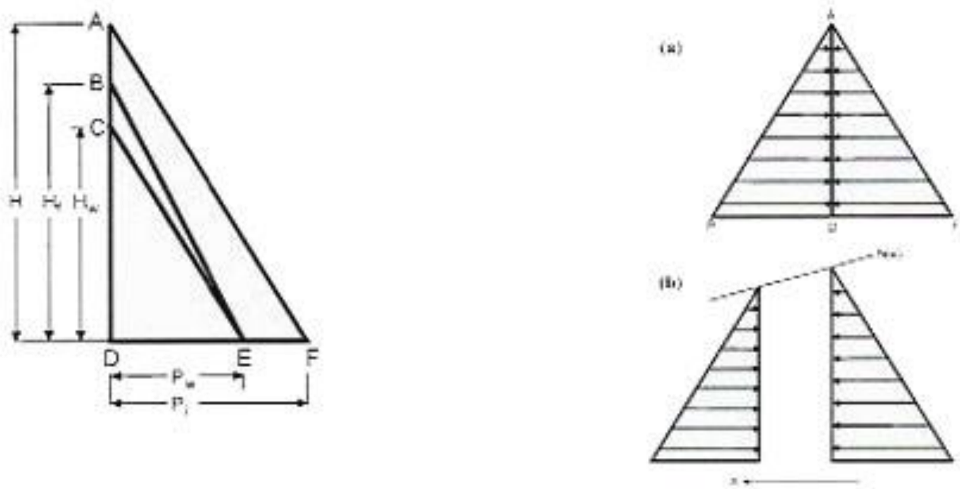

Figure 3: Figure 3 (left) and Figure 4 (right) from Van der Veen (2016).