Supplementary material to 'Boundary layer models for calving marine outlet glaciers'

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March 16, 2017

1 Overview

In this supplementary text we provide the following:

1. A derivation of a thickness condition for the calving cliff when calving is happening in the Nick *et al.* (2010) calving model (section 2 below)

2. A demonstration of the self-similarity of the boundary layer model that leads to a reduced form of the flux law, equation (10a) in the main text (section 3)

3. An analysis of the boundary layer model, demonstrating that flux is uniquely determined by flotation thickness at the grounding line, channel width, basal drag coefficient and calving parameter, and a sketch of the numerical method used for the computations (section 4; the code used is supplied seperately as part of the supplementary material package)

4. A detailed analysis of the approximate solutions presented in sections 5.1–5.4 of the main text, showing that they are asymptotic solutions in appropriate limits rather than uncontrolled *ad hoc* approximations (section 5)

The code used for all computations in the main paper is supplied separately, see the appropriate README files in the code packages.

2 The CD calving model

Here we analyse the calving model due to Nick *et al.* (2010) and reduce it to a condition for ice thickness given by, in the notation of our paper,

$$\text{either } h = h_c \qquad \text{at } x = x_c \text{ if } \dot{x}_c \leq u(x_c), \tag{1}$$
$$\text{or } \dot{x}_c = u \qquad \text{at } x = x_c \text{ if } h > h_c, \tag{2}$$

where h_c is

$$h_c = -b\phi\left(-d_w/b\right) \tag{3}$$

$$\text{where } \phi\left(-d_w/b\right) = \begin{cases} 2(\rho_w/\rho_i)(-d_w/b) & -d_w/b < 1/2 \\ \nu + \sqrt{\nu^2 - (\rho_w/\rho_i)} & -d_w/b \geq 1/2, \end{cases} \tag{4}$$

We focus on what they term the CD model for tensile failure, in which calving occurs when the combined depth d_s of surface crevasses and d_b of basal crevasses equals the ice thickness h. Only the essential components of the model are reiterated here and a detailed derivation may be found in Nick *et al.* (2010).

The construction of the model assumes that the 'non-cryostatic' contribution to normal stress acting on a vertical plane near the calving front (this stress being termed the 'resistive stress' R_{xx} in their paper, and the 'extensional stress' Σ in ours) is given by $\Sigma = R_{xx} = 2\bar{B}|u_x|^{1/n-1}u_x$ and is therefore independent of depth. In other words, the usual (compressive negative) normal stress on a vertical plane in the ice is given by

$$\sigma_{xx} = R_{xx} - \rho_i g(h - z).$$

where ρ_i is the density of water and g is acceleration due to gravity, while z is elevation relative to the base of the ice. This formula is appropriate for rapidly moving ice with insignificant shearing in the vertical (e.g. Schoof & Hindmarsh, 2010).

Nick *et al.* (2010) assume that crevasses penetrate to the distances from the upper and lower boundaries at which compressive normal stress $-\sigma_{xx}$ is larger than the water pressure pressing on the walls of a crack. If water depth in the crack is d_w, the water pressure is $p_{w,s} = \rho_w g d_w$. Denote by $d_s = h - z$ the depth below the ice surface of the crack tip. Then $p_{w,s} = -\sigma_{xx}$ implies

$$\rho_i g d_s - R_{xx} = \rho_w g d_w$$

This allows the penetration depth for surface crevasses to be computed as

$$d_s = \frac{R_{xx}}{\rho_i g} + \frac{\rho_w}{\rho_i} d_w, \tag{5}$$

A similar calculation is applied to finding the height d_b from the base of the ice to which basal crevasses penetrate. Again, water presses against the sides of the crack. At the crack tip, that pressure is $p_{w,b} = \rho_w g(h_b - d_b)$ where h_b is the depth of the base of the ice below sea level. How h_b relates to depth to bedrock and ice thickness depends on whether the ice grounded or afloat: we have

$$h_b = \begin{cases} (\rho_i/\rho_w)h & \text{if } h < -(\rho_w/\rho_i)b, \\ -b & \text{if } h \geq -(\rho_w/\rho_i)b, \end{cases} \tag{6}$$

the first option corresponding to floating ice and the second to grounded ice. Equating $-\sigma_{xx}$ with $p_{w,b}$ gives

$$\rho_i g(h - d_b) - R_{xx} = \rho_w g(h_b - d_b)$$

or

$$d_b = d_b = \frac{R_{xx}}{\rho_w - \rho_i} - (\rho_i g h - \rho_w g h_b) \tag{7}$$

Note that the expression in brackets on the right (denoted by h_{ab} by Nick *et al.* (2010)) vanishes for a floating terminus, and is positive for a fully grounded terminus, when $-b < (\rho_i/\rho_w)h$.

At the ice front, the extensional stress is given in terms of ice and bed geometry by equation (1e) of the main text as

$$R_{xx} = \begin{cases} \frac{\rho_i g}{2}\left(1 - \frac{\rho_i}{\rho_w}\right) h & \text{if } h < -(\rho_w/\rho_i)b, \\ \frac{\rho_i g}{2}\left(h - \frac{\rho_w}{\rho_i}\frac{b^2}{h}\right) & \text{if } h \geq -(\rho_w/\rho_i)b. \end{cases} \tag{8}$$

It is now a matter of simple algebra to solve for $h = h_c$ at a glacier terminus where calving is actively occurring. We have to distinguish the case of a floating and grounded calving front. For a floating calving front,

$$d_s + d_b = h/2 + \rho_w d_w/\rho_i$$

at the calving front. Calving occurs when

$$d_s + d_b = h = h_c,$$

so the crevasses together penetrate through the full thickness of ice. This occurs when

$$h = \frac{2\rho_w}{\rho_i} d_w < -\frac{\rho_w}{\rho_i} b : \tag{9}$$

for a floating calving front, ice thickness is simply proportional to the crevasse water depth d_w, with no other parameter apart from the density ratio ρ_w/ρ_i. For a grounded calving face with $h_c > -b$, we have

$$d_s + d_b = \frac{\rho_w}{\rho_w - \rho_i}\left[-b - \frac{\rho_w}{2\rho_i}\frac{b^2}{h_c} - \left(\frac{\rho_i}{\rho_w} - \frac{1}{2}\right) h\right] + \frac{\rho_w d_w}{\rho_i}.$$

Hence $d_s + d_b = h_c$ leads to a quadratic equation for h_c:

$$\left(\frac{h_c}{-b}\right)^2 - 2\left[1 + \left(\frac{\rho_w}{\rho_i} - 1\right)\frac{d_w}{-b}\right]\frac{h_c}{-b} + \frac{\rho_w}{\rho_i} = 0$$

with solution

$$h_c = -b\left[1 + \left(\frac{\rho_w}{\rho_i} - 1\right)\frac{d_w}{-b} \pm \sqrt{\left(1 + \left(\frac{\rho_w}{\rho_i} - 1\right)\frac{d_w}{-b}\right)^2 - \frac{\rho_w}{\rho_i}}\right]. \tag{10}$$

where we require $h_c > -(\rho_w/\rho_i)b$ in order for this formula to hold.

Importantly, the calving front needs to be stable to catastrophic, calving-driven retreat: if the front is perturbed in an upstream direction into thicker ice, calving should cease, allowing the front to advance and the ice to thin as it does. If this is not the case, then progressively more calving will result as ice breaks off to reveal a taller calving face, at which crevasses will again penetrate through the full thickness of the ice (this unstable situation is analogous to the 'ice cliff collapse' scenario described in Pollard *et al.* (2015), only that the model in Nick *et al.* (2010) does not provide for a time scale, and implicitly assumes the cliff collapse happens infinitely fast).

Consider an ice front that is perturbed slightly upstream from its calving position into thicker ice. We require that the total crevasse depth $d_s + d_b$ there is less than the new, slightly larger, ice front thickness, so that calving ceases and the ice front moves forward again and thins until calving recommences. This basic requirement for stability can therefore be written mathematically as

$$\frac{\mathrm{d}(d_s + d_b)}{\mathrm{d}h_c} < 1. \tag{11}$$

It is easy to show that the solution in (9) satisfies this criterion, while only the *larger* of the two roots in (10) also does. Moreover, the smaller root also violates the assumption that the ice front is actually grounded, since $\phi < r^{-1}$ at that root. (We can also note that the shear failure model of Bassis & Walker (2011) violates this stability constraint.)

From the formulae (9) and (10) combined with the inequality constraints on h_c, it is then clear that the value $d_w/(-b)$ of the ratio crevasse water depth to the depth to sea floor uniquely determines whether the corresponding calving front is floating or grounded. More precisely, assume we know that a stably calving glacier terminus exists at a position x_f, where depth to the sea floor is $-b(x_f)$. The thickness of the terminus can then be written in the form of a fraction of water depth $-b$

$$h_c = -b\phi\left(-d_w/b\right), \tag{12}$$

where the fraction depends only on $d_w/(-b)$ as stated in (4).

Note that this condition holds only at calving fronts where calving is occurring, in which case the velocity of ice relative to the migration of velocity of the calving front needs to point out of the domain, so $\mathrm{d}x_c/\mathrm{d}t - u(x_c) > 0$. The alternative is that there is no calving, so h is larger than the critical value h_c given by (12), and the calving front moves at the velocity of the ice, $\mathrm{d}x_c/\mathrm{d}t = u(x_c)$.

3 A grounding line flux formula

Here we show that the the relationship between flux, geometry and other model parameters must take the form

$$Q = WH_f^{n+1}G_\Lambda\left(\frac{H_f}{\Lambda}, \frac{\Gamma W^{(nm+n+m+1)/(n+1)}}{\Lambda^{2-nm}}, n, m, r\right). \tag{13a}$$

or simpler still as

$$Q = W H_f^{n+1} G_\Gamma \left(\frac{\Gamma W^{(nm+n+m+1)/(n+1)}}{H_f^{2-nm}}, n, m, r \right). \tag{13b}$$

for calving at flotation.

For reference, the boundary layer model as stated in the main text is

$$\begin{aligned} 4(H|U_X|^{1/n-1}U_X)_X - W^{-1/n-1}H|U|^{1/n-1}U \\ -\Gamma\theta|U|^{m-1}U - [1-(1-\theta)r]\, HH_X = 0, && (14a) \\ (HU)_X = 0, && (14b) \end{aligned}$$

for $X < X_c$ where

$$\theta = 1 \qquad \text{for } H \geq H_f, \qquad \theta = 0 \qquad \text{otherwise.} \tag{14c}$$

The additional boundary condition at the calving front takes the form

$$\begin{aligned} 4H|U_X|^{1/n-1}U_X = (1-(1-\theta)r)H^2/2 - \theta r H_f^2/2 \qquad \text{at } X = X_c, && (14d) \\ H = rH_f\phi\left(\Lambda H_f^{-1}\right) \qquad \text{at } X = X_c && (14e) \end{aligned}$$

and the matching conditions are

$$\lim_{X\to-\infty} UH = Q = \lim_{x\to x_g^-}(-w^{n+1}h|h_x|^{1/m-1}h_x), \qquad W^{-1/n-1}Q|U|^{1/n-1} \sim -(Q/U)(Q/U)_X,$$

$$U \to 0 \qquad \text{as} \qquad X \to -\infty. \tag{14f}$$

We begin by using a scale invariance in the boundary layer model (14). Rescale the model by putting

$$U = Q^{n/(n+1)}W^{1/(n+1)}\mathcal{U}, \qquad H = Q^{1/(n+1)}W^{-1/(n+1)}\mathcal{H}, \qquad X = W\mathcal{X} \tag{15}$$

and similarly

$$\begin{aligned} \mathcal{C} = \Gamma W^{(m+n+3)/(n+1)}/Q^{(2-nm)/(n+1)}, \\ \mathcal{H}_f = H_f(W/Q)^{1/(n+1)}, \qquad \mathcal{L} = \Lambda(W/Q)^{1/(n+1)}, \qquad X_c = W\mathcal{X}_c. && (16) \end{aligned}$$

With these definitions, it is easy to show that (14) becomes

$$\begin{aligned} 4(\mathcal{H}|\mathcal{U}_\mathcal{X}|^{1/n-1}\mathcal{U}_\mathcal{X})_\mathcal{X} - \mathcal{H}|\mathcal{U}|^{1/n-1}\mathcal{U} - \theta\mathcal{C}|\mathcal{U}|^{m-1}\mathcal{U} \\ - [1-(1-\theta)r]\,\mathcal{H}\mathcal{H}_\mathcal{X} = 0 \quad \text{for } \mathcal{X} < \mathcal{X}_c, && (17a) \\ (\mathcal{H}\mathcal{U})_\mathcal{X} = 0 \quad \text{for } \mathcal{X} < \mathcal{X}_c, && (17b) \end{aligned}$$

where

$$\theta = \begin{cases} 1 & \text{for } \mathcal{H} \geq \mathcal{H}_f, \\ 0 & \text{otherwise.} \end{cases} \tag{17c}$$

and the far-field conditions become for $\mathcal{X} \to -\infty$

$$\mathcal{U}\mathcal{H} \sim 1 \qquad |\mathcal{U}|^{1/n-1} \sim \mathcal{U}^{-3}\mathcal{U}_{\mathcal{X}}, \qquad \mathcal{U} \to 0. \tag{17d}$$

while at the calving front

$$4\mathcal{H}|\mathcal{U}_{\mathcal{X}}|^{1/n-1}\mathcal{U}_{\mathcal{X}} = (1-(1-\theta)r)\mathcal{H}^2/2 - \theta r \mathcal{H}_f^2/2 \qquad \text{at } \mathcal{X} = \mathcal{X}_c, \tag{17e}$$

$$\mathcal{H} = r\mathcal{H}_f \phi(\mathcal{L}\mathcal{H}_f^{-1}) \qquad \text{at } \mathcal{X} = \mathcal{X}_c \tag{17f}$$

This takes the same form as the original boundary layer problem (14), but with all variables represented by their calligraphic font counterparts, and flux as well as channel width replaced by unity. We show below in 4 the rescaled problem (17) indeed has a solution if and only if its parameters satisfy a relationship of the form

$$F(r, n, m, \mathcal{H}_f, \mathcal{L}, \mathcal{C}) = 0, \tag{18}$$

the form of F being determined from the form of certain orbits of a dynamical system that can be computed numerically.

Assume the CD calving law of Nick *et al.* (2010) holds, where ϕ is given by (4). We can then assume that F can be inverted to write, for some function G_Λ,

$$\mathcal{H}_f^{-(n+1)} = G_\Lambda\left(\frac{\mathcal{H}_f}{\mathcal{L}}, \frac{\mathcal{C}}{\mathcal{L}^{2-nm}}, n, m, r\right).$$

With the definitions of $\mathcal{H}_f$, $\mathcal{L}$ and $\mathcal{C}$, this implies that flux can be written in the form of equation (10a) in the main text, with the function G_Λ determined by the same procedure as F (see section 4).

For the simpler case of 'calving at flotation' ($\phi = r^{-1}$), F cannot depend on $\mathcal{L}$, and we can alternatively rewrite (18) in the form

$$\mathcal{H}_f^{-(n+1)} = G_\Gamma\left(\frac{\mathcal{C}}{\mathcal{H}_f^{2-nm}}, n, m, r\right),$$

where G_Γ is a different function, determined by the same integration procedure as F. When substituting the definitions of $\mathcal{H}_f$ and $\mathcal{C}$, we obtain (10a) of the main text. An even simpler case arises when there is additionally no basal drag, so $\Gamma = 0$: in that case, we get a simple power law relationship between flux and channel geometry at the terminus:

$$Q = G_\Gamma(0, n, m, r) W H_f^{n+1}. \tag{19}$$

This is in fact analogous to a result derived for flux at ice shelf fronts due to Hindmarsh (2012) and reiterated by Pegler (2016).

4 The boundary layer problem as a dynamical system

In this section, we demonstrate how the function F in (18) can be constructed, allowing (18) to be solved computationally for flux Q. We closely follow the analysis used in appendix A of Schoof (2011), using a change of variables to obtain from (17) a non-singular system that is amenable to both analysis and a reasonably straightforward numerical solution. Let

$$\mathcal{Q} = \mathcal{U}\mathcal{H}, \qquad \Psi = \mathcal{Q}^{-1}\mathcal{U}^{-(2n+1)/n^2}|\mathcal{U}_{\mathcal{X}}|^{1/n-1}\mathcal{U}_{\mathcal{X}}, \qquad \xi = \mathcal{U}^{(n+1)^2/n^2}. \tag{20}$$

where $\mathcal{U}$, $\mathcal{H}$ and $\mathcal{X}$ are defined in §3. Under this transformation, (14) can be written as a three-dimensional first-order system for $-\infty < \mathcal{X} < \mathcal{X}_c$:

$$\xi_{\mathcal{X}} = \alpha(n+1)^2 n^{-2}\xi^2\mathcal{Q}^{n+1}, \tag{21a}$$

$$\Psi_{\mathcal{X}} = \alpha\Big[4^{-1}\Psi^{-n} - 4^{-1}\mathcal{Q}^{n+1}[1-(1-\theta)r] - (2n+1-n^2)n^{-2}\mathcal{Q}^{n+1}\xi\Psi + 4^{-1}\mathcal{C}\theta\mathcal{Q}^{-1}\xi^{(n^2(m+1)-n)/(n+1)^2}\Psi^{-n}\Big], \tag{21b}$$

$$\mathcal{Q}_{\mathcal{X}} = 0, \tag{21c}$$

where

$$\alpha = \mathcal{Q}^{-1}\Psi^n\xi^{-1/(n+1)} \qquad \text{and} \qquad \theta = \begin{cases} 1 & \text{if } \mathcal{Q}\xi^{-n^2/(n+1)^2} > \mathcal{H}_f, \\ 0 & \text{otherwise,} \end{cases} \tag{21d}$$

and we seek orbits that satisfy the transformed matching conditions (14f)

$$(\Psi, \xi, \mathcal{Q}) \to (1, 0, 1) \qquad \text{as } \mathcal{X} \to -\infty, \tag{21e}$$

and the boundary conditions (14d)–(14e)

$$\Psi\xi = [1-(1-\theta)r]/8 - \theta r\mathcal{H}_f^2\mathcal{Q}^{-2}\xi^{2n^2/(n+1)^2}/8 \quad \text{at } \mathcal{X} = \mathcal{X}_c, \tag{21f}$$

$$\mathcal{Q}\xi^{-n^2/(n+1)^2} = r\mathcal{H}_f\phi(\mathcal{L}\mathcal{H}_f^{-1}) \quad \text{at } \mathcal{X} = \mathcal{X}_c. \tag{21g}$$

We will demonstrate that an orbit of $(\Psi, \xi, \mathcal{Q})$ satisfying all the conditions at $\mathcal{X} = -\infty$ and $\mathcal{X} = \mathcal{X}_c$ requires a relationship of the form (18) holds. As in Schoof (2011), we can absorb α into the independent variable by defining

$$\zeta = \int_0^{\mathcal{X}} \alpha(\mathcal{Q}(\mathcal{X}'), \Psi(\mathcal{X}'), \xi(\mathcal{X}'))\mathrm{d}\mathcal{X}',$$

and we obtain a non-singular dynamical system with independent variable ζ; the system is however non-smooth on account of the discontinuity in θ.

The orbit that solves our problem must first satisfy the matching conditions (21e), meaning it emerges from the fixed point $(\Psi, \xi, \mathcal{Q}) = (1, 0, 1)$, for which $\theta = 1$. This orbit is unique by the same argument as in Schoof (2011): we are restricted to the $\mathcal{Q} = 1$ plane, and we can show that, in that plane, the fixed point has a stable manifold and an unstable centre manifold, which ensures that the latter is unique (Sijbrand, 1985) and can be identified with the required orbit. There is a slight technical difficulty when $m < 1/n + 2(n+1)^2/n^2 - 1$, in which case (21b) is not twice continuously differentiable near the the fixed point; a further coordinate transform $\nu = \xi^{(n^2(m+1)-n)/(2(n+1)^2)}$ then yields the requisite degree of smoothness near the corresponding fixed point in the (Ψ, ν)-plane, and the result follows.

A grounded terminus corresponds to (21f) and (21g) holding with $\theta = 1$. For a given set of parameters n, m and $\mathcal{C}$, integrate the orbit until it intersects the curve $\Psi\xi = 1/8 - r\mathcal{H}_f^2\xi^{2n^2/(n+1)^2}/8$. The point of intersection defines a value ξ_c, and the value $\mathcal{H}_f$ then needs to satisfy (21g) with $\xi = \xi_c$, so we have

$$F(r, n, m, \mathcal{H}_f, \mathcal{L}, \mathcal{C}) = \xi_c(n, m, r, \mathcal{H}_f, \mathcal{C})^{-n^2/(n+1)^2} - r\mathcal{H}_f\phi(\mathcal{L}\mathcal{H}_f^{-1}) = 0 \qquad (22)$$

and are looking for a solution of $F = 0$. Note that this procedure is effectively the same as that described in appendix A of Schoof (2011). The computation is self-consistent so long as, at the solution, the calving front is indeed grounded, meaning $r\phi(\mathcal{L}\mathcal{H}_f^{-1}) \geq 1$

Alternatively, consider a terminus that is afloat, so $r\phi(\mathcal{L}\mathcal{H}_f^{-1}) < 1$. The procedure for finding F is essentially the same as above. We can again follow the orbit out of the fixed point $(\Psi, \xi, \mathcal{Q}) = (1, 0, 1)$ as described above. When $\xi = \mathcal{H}_f^{-(n+1)^2/n^2}$, θ changes discontinuously, corresponding to floatation. The orbit can be integrated up to the curve $\Psi\xi = (1 - r)/8$, defining a point $(\Psi_f, \xi_f, \mathcal{Q}_f)$ depending on $(n, m, r, \mathcal{H}_f, \mathcal{C})$. The function F in (18) is then defined by imposing (21g):

$$F(r, n, m, \mathcal{H}_f, \mathcal{L}, \mathcal{C}) = \\ \mathcal{Q}_f(n, m, r, \mathcal{H}_f, \mathcal{C})\xi_f(n, m, r, \mathcal{H}_f, \mathcal{C})^{-n^2/(n+1)^2} - r\mathcal{H}_f\phi(\mathcal{L}\mathcal{H}_f^{-1}) = 0. \qquad (23)$$

In each of the cases above, the function F needs to be established through integration of the dynamical system. We need to solve $F = 0$ for Q as a function of H_f, W, and the remaining non-geometrical parameters, which is done here by means of a shooting method. The code used in these computations is also included as supplementary material with this paper.

5 Approximate boundary layer solutions

5.1 Calving at or near flotation

Here we show how approximate solutions to the boundary layer problem (14) can be derived formally. These approximate solutions are then used in the main text of the paper to shed light on the physical mechanisms by which flux through the grounding line is controlled. In this section, we explore the origins of the anomalous flux-flotation-thickness relationship.

Let $\delta = 1 - r$. In practice, $\delta \approx 0.1$ is small, and we can use this fact by analogy with similar work in appendix A in Schoof (2007). It is convenient to work directly with the version (17) of the boundary layer. We consider for now only the grounded part of the glacier with $\theta = 1$, and rescale

$$\tilde{\mathcal{X}} = \delta^{n/(n+1)}\mathcal{X}, \qquad \tilde{\mathcal{U}} = \delta^{-n^2/(n+1)^2}\mathcal{U}, \qquad \tilde{\mathcal{H}} = \delta^{n^2/(n+1)^2}\mathcal{H} \qquad (24)$$

At leading order in δ, this results in

$$-\tilde{\mathcal{H}}|\tilde{\mathcal{U}}|^{1/n-1}\tilde{\mathcal{U}} - \tilde{\mathcal{C}}|\tilde{\mathcal{U}}|^{m-1}\tilde{\mathcal{U}} - \tilde{\mathcal{H}}\tilde{\mathcal{H}}_{\tilde{\mathcal{X}}} = 0 \qquad \text{for } \tilde{\mathcal{X}} < 0, \tag{25a}$$

$$(\tilde{\mathcal{H}}\tilde{\mathcal{U}})_{\tilde{\mathcal{X}}} = 0 \qquad \text{for } \tilde{\mathcal{X}} < 0, \tag{25b}$$

$$\tilde{\mathcal{U}}\tilde{\mathcal{H}} \sim 1 \qquad |\tilde{\mathcal{U}}|^{1/n-1} \sim \tilde{\mathcal{U}}^{-3}\tilde{\mathcal{U}}_{\tilde{\mathcal{X}}}, \qquad \tilde{\mathcal{U}} \to 0 \qquad \text{as } \tilde{\mathcal{X}} \to -\infty. \tag{25c}$$

Here, $\tilde{\mathcal{C}} = \delta^{(n+nm-1)n/(n+1)^2}\mathcal{C}$, and we assume that $\tilde{\mathcal{C}} = O(1)$ to permit for significant basal drag, while still ensuring that lateral drag plays a leading order role. We consider the case of dominant basal drag below, in section 5.2.

We also require the relevant boundary conditions to hold at the grounding line $\tilde{\mathcal{X}} = 0$. Write the critical flotation thickness $2\mathcal{L}$ at which the calving front is *at* flotation in the form $2\mathcal{L} = \delta^{-n^2/(n+1)^2}\tilde{\mathcal{H}}_{f0}$ and suppose the actual flotation thickness is a small perturbation away from the critical value, $\mathcal{H}_f = \delta^{-n^2/(n+1)^2}\left[\tilde{\mathcal{H}}_{f0} + \delta\tilde{\mathcal{H}}'_f\right]$. To leading order, boundary conditions at the grounding line $\tilde{\mathcal{X}} = 0$ can then be written as

$$\tilde{\mathcal{H}} = \tilde{\mathcal{H}}_{f0}, \qquad 4|\tilde{\mathcal{U}}_{\tilde{\mathcal{X}}}|^{1/n-1}\tilde{\mathcal{U}}_{\tilde{\mathcal{X}}} = \tilde{\mathcal{T}},$$

where $\tilde{\mathcal{T}}$ is an extensional stress at the grounding line. For a calving cliff above flotation with $\tilde{\mathcal{H}}'_f < 0$, this can be written by linearizing ϕ as is done in the main text,

$$\tilde{\mathcal{T}} = \frac{1}{2}\tilde{\mathcal{H}}_{f0} - \tilde{\mathcal{H}}'_f. \tag{26}$$

When there is a floating ice tongue, the stress $\tilde{\mathcal{T}}$ at the grounding line needs to be solved for using a model for ice flow in the ice tongue, which we deal with below.

We confine ourselves to the special case $m = 1/n$, for which an analytical solution for $\tilde{\mathcal{H}}\tilde{\mathcal{U}}$ is easy to find in terms of $\tilde{\mathcal{T}}$. Since the matching conditions dictate that $\tilde{\mathcal{H}}\tilde{\mathcal{U}} \equiv 1$, we get

$$1 = \tilde{\mathcal{H}}\tilde{\mathcal{U}} = \frac{\tilde{\mathcal{H}}_{f0}^{(3n+1)/(n+1)}\tilde{\mathcal{T}}^{n^2/(n+1)}}{4^{n^2/(n+1)}(\tilde{\mathcal{C}} + \tilde{\mathcal{H}}_{f0})^{n/(n+1)}}, \tag{27}$$

for the case of flotation thickness slightly below the critical value, , when $\tilde{\mathcal{T}}$ is given by (26), using the definitions of $\tilde{\mathcal{H}}_{f0}$, $\tilde{\mathcal{H}}_f$ and $\tilde{\mathcal{C}}$ can be used to turn (27) into equation (16) of the main text:

$$Q \approx \left(\frac{1-r}{8}\right)^{n^2/(n+1)} \frac{H_{f0}^{(3n+1)/(n+1)}\left[H_{f0} - 2(1-r)^{-1}H'_f\right]^{n^2/(n+1)}}{\left(\Gamma + W^{-(n+1)/n}H_{f0}\right)^{n/(n+1)}}$$

For calving at flotation, it suffices to put $\tilde{\mathcal{H}}_{f0} = \tilde{\mathcal{H}}_f$, $\tilde{\mathcal{H}}'_f = 0$ (which gives the correct stress in (26) for a calving front at flotation) and use the definitions of $\tilde{\mathcal{H}}_f$ and $\tilde{\mathcal{C}}$ to derive equation (12) of the main text

$$Q \approx \left(\frac{1-r}{8}\right)^{n^2/(n+1)} \frac{H_f^{(n^2+3n+1)/(n+1)}}{\left(\Gamma + W^{-(n+1)/n}H_f\right)^{n/(n+1)}},$$

When there is a floating tongue attached, $\tilde{\mathcal{T}}$ needs to be determined from a model for flow in the floating tongue, where $\theta = 0$. Because of the much smaller driving stress in the floating tongue, we require a different rescaling for distance from that in (24): We put

$$\breve{\mathcal{X}} = \delta^{-1}\tilde{\mathcal{X}},$$
$$\breve{\mathcal{U}} = \tilde{\mathcal{U}}, \qquad \breve{\mathcal{H}} = \tilde{\mathcal{H}}, \tag{28}$$

and define $\breve{\mathcal{T}}$ through

$$\breve{\mathcal{U}}_{\breve{\mathcal{X}}} = 4^{-n}\delta|\breve{\mathcal{T}}|^{n-1}\breve{\mathcal{T}}. \tag{29a}$$

The rescaled equations (17) for the ice tongue become

$$(\breve{\mathcal{H}}\breve{\mathcal{T}})_{\breve{\mathcal{X}}} - \breve{\mathcal{H}}|\breve{\mathcal{U}}|^{1/n-1}\breve{\mathcal{U}} - \breve{\mathcal{H}}\breve{\mathcal{H}}_{\breve{\mathcal{X}}} = 0, \tag{29b}$$
$$\left(\breve{\mathcal{H}}\breve{\mathcal{U}}\right)_{\breve{\mathcal{X}}} = 0 \tag{29c}$$

for $0 < \breve{\mathcal{X}} < \breve{\mathcal{X}}_c$, with boundary conditions at the calving front $\breve{\mathcal{X}} = \breve{\mathcal{X}}_c$ and at the grounding line $\breve{\mathcal{X}} = 0$

$$\breve{\mathcal{T}} = \breve{\mathcal{H}}/2 \qquad \text{at } \breve{\mathcal{X}} = \breve{\mathcal{X}}_c, \tag{30a}$$
$$\breve{\mathcal{H}} = \tilde{\mathcal{H}}_{f0}, \qquad \text{at } \breve{\mathcal{X}} = \breve{\mathcal{X}}_c, \tag{30b}$$
$$\tilde{\mathcal{U}} = \breve{\mathcal{U}}, \qquad \tilde{\mathcal{T}} = \breve{\mathcal{T}}, \qquad \breve{\mathcal{H}} = \tilde{\mathcal{H}}_{f0} + \delta\tilde{\mathcal{H}}'_f \qquad \text{at } \breve{\mathcal{X}} = \tilde{\mathcal{X}} = 0. \tag{30c}$$

The quantity of primary interest in the ice tongue is the extensional stress $\breve{\mathcal{T}}$, as this couples to the grounded ice flow to determine ice flux at the grounding line through (27). In order to determine $\breve{\mathcal{T}}$, we need to find the length of the ice tongue, as this determines the net buttressing effect of lateral shear. To find that length, we have to expand ice thickness to first order,

$$\breve{\mathcal{T}} = \breve{\mathcal{T}}^{(0)} + O(\delta), \qquad \breve{\mathcal{U}} = \breve{\mathcal{U}}^{(0)} + O(\delta), \qquad \breve{\mathcal{H}} = \breve{\mathcal{H}}^{(0)} + \delta\breve{\mathcal{H}}^{(1)} + O(\delta^2).$$

At leading order, we obtain from (29) and (30)

$$\breve{\mathcal{U}}^{(0)} \equiv \lim_{\tilde{\mathcal{X}}\to 0^-} \tilde{\mathcal{U}} = \text{constant} \qquad \breve{\mathcal{H}}^{(0)} \equiv \tilde{\mathcal{H}}_{f0} = \text{constant}$$

throughout the ice tongue: ice velocity and thickness are constant at leading order. In addition,

$$\breve{\mathcal{H}}^{(0)}\breve{\mathcal{T}}^{(0)}_{\breve{\mathcal{X}}} - \breve{\mathcal{H}}^{(0)}\breve{\mathcal{U}}^{(0)\,1/n} = 0, \tag{31a}$$

while at first order

$$\breve{\mathcal{U}}^{(0)}\breve{\mathcal{H}}^{(1)}_{\breve{\mathcal{X}}} = -\breve{\mathcal{H}}^{(0)}|\breve{\mathcal{T}}^{(0)}|^{n-1}\breve{\mathcal{T}}^{(0)} \tag{31b}$$

with boundary conditions

$$\breve{\mathcal{H}}^{(1)}(0) = \breve{\mathcal{H}}'_f, \qquad \breve{\mathcal{H}}^{(1)}(\breve{\mathcal{X}}_c) = 0, \qquad \breve{\mathcal{T}}^{(0)}(\breve{\mathcal{X}}_c) = \breve{\mathcal{H}}_{f0}/8 \tag{31c}$$

We have a constant leading-order velocity $\breve{\mathcal{U}}^{(0)}$ and can use this to find

$$\breve{\mathcal{T}}(\breve{\mathcal{X}}) = \breve{\mathcal{H}}_{f0}/2 - \breve{\mathcal{U}}^{(0)\,1/n}(\breve{\mathcal{X}}_c - \breve{\mathcal{X}})$$

so the required extensional stress at the grounding line is

$$\breve{\mathcal{T}}(0) = \breve{\mathcal{H}}_{f0}/2 - \breve{\mathcal{U}}^{(0)\,1/n}\breve{\mathcal{X}}_c \tag{32}$$

Solving for $\breve{\mathcal{H}}^{(1)}$ yields

$$\breve{\mathcal{U}}^{(0)}\breve{\mathcal{H}}^{(1)}(\breve{\mathcal{X}}) = \breve{\mathcal{U}}^{(0)}\breve{\mathcal{H}}_f' - \frac{4\breve{\mathcal{H}}_{f0}}{(n+1)\breve{\mathcal{U}}^{(0)\,1/n}}\left[\breve{\mathcal{H}}_{f0}/8 - \breve{\mathcal{U}}^{(0)\,1/n}(\breve{\mathcal{X}}_c - \breve{\mathcal{X}})/4\right]^{n+1}$$

and hence $(31\text{c})_2$ yields the required constraint on $\breve{\mathcal{X}}_c$

$$\left[\frac{\breve{\mathcal{H}}_{g0}^{n+1}}{8^{n+1}} - \frac{(n+1)\breve{\mathcal{U}}^{(0)\,1/n}\breve{\mathcal{U}}^{(0)}\breve{\mathcal{H}}_f'}{4\breve{\mathcal{H}}_{f0}}\right]^{1/(n+1)} = \breve{\mathcal{H}}_{f0}/8 - \breve{\mathcal{U}}^{(0)\,1/n}\breve{\mathcal{X}}_c/4 = \breve{\mathcal{T}}(0) \tag{33}$$

Given $\tilde{\mathcal{T}} = \breve{\mathcal{T}}^{(0)}(\breve{\mathcal{X}} = 0)$, we get from (27)

$$1 = \frac{\breve{\mathcal{H}}_{f0}^{(3n+1)/n}}{\tilde{\mathcal{C}} + \breve{\mathcal{H}}_{g0}}\left[\frac{\breve{\mathcal{H}}_{f0}^{n+1}}{8^{n+1}} - \frac{(n+1)\breve{\mathcal{H}}_f'}{4\breve{\mathcal{H}}_{f0}^{(2n+1)/n}}\right]^{n/(n+1)} \tag{34}$$

Substituting the definitions of $\breve{\mathcal{H}}_{f0}$ and $\breve{\mathcal{H}}_f'$, we obtain

$$Q \approx \frac{H_{f0}^{(3n+1)/(n+1)}}{\left(\Gamma + W^{-(n+1)/n}H_{f0}\right)^{n/(n+1)}}\left\{\left[\frac{(1-r)H_{f0}}{8}\right]^{n+1} - \frac{(n+1)Q^{(n+1)/n}H_f'}{W^{(n+1)/n}H_{f0}^{(2n+1)/n}}\right\}^{n^2/(n+1)^2}$$

5.2 Large basal friction

The previous section justifies an anomalous flux-flotation-thickness relationship for the CD calving model. For a grounded calving front, this anomalous behaviour persists even in the limit of large basal friction. It arises purely because the calving cliff protrudes significantly above the flotation level, and causes a sharp increase in extensional stress. Mathematically, the denominator of (27) is simply dominated by $\tilde{\mathcal{C}}^{n/(n+1)}$, but the formula (27) still holds. When the calving front is afloat, the anomalous relationship no longer applies in the limit of large basal friction coefficient (that is, in the limit of a large $\tilde{\mathcal{C}}$) because our floating ice tongue model (29) is no longer appropriate. We can justify this formally by recognizing that another rescaling is necessary.

It is easy to see that $\breve{\mathcal{T}} \sim \breve{\mathcal{H}}$ and $\breve{\mathcal{U}} \sim \breve{\mathcal{H}}^{2n/(n+1)}\breve{\mathcal{T}}^{n^2/(n+1)}/\tilde{\mathcal{C}}^{n/(n+1)}$. With the constraint that $\breve{\mathcal{H}}\breve{\mathcal{U}} \sim 1$, this requires the rescaling

$$\breve{\mathcal{H}} \sim \tilde{\mathcal{C}}^{n/(n^2+3n+1)}\breve{\mathcal{H}}, \qquad \breve{\mathcal{U}} \sim \tilde{\mathcal{C}}^{-n/(n^2+3n+1)}\breve{\mathcal{U}}, \qquad \breve{\mathcal{T}} \sim \tilde{\mathcal{C}}^{n/(n^2+3n+1)}\breve{\mathcal{T}}, \tag{35}$$

and, in the floating ice tongue,

$$\breve{\mathcal{X}} = \tilde{\mathcal{C}}^{(n+1)/(n^2+3n+1)}\breve{\mathcal{X}}.$$

The ice tongue model (29) then becomes

$$(\check{\mathcal{H}}\check{\mathcal{T}})_{\check{\mathcal{X}}} - \check{\mathcal{H}}|\check{\mathcal{U}}|^{1/n-1}\check{\mathcal{U}} - \check{\mathcal{H}}\check{\mathcal{H}}_{\check{\mathcal{X}}} = 0 \tag{36a}$$

$$\left(\check{\mathcal{H}}\check{\mathcal{U}}\right)_{\check{\mathcal{X}}} = 0 \tag{36b}$$

$$\check{\mathcal{U}}_{\check{\mathcal{X}}} = 4^{-n}\tilde{\mathcal{C}}^{(n+1)^2/(n^2+3n+1)}\delta|\check{\mathcal{T}}|^{n-1}\check{\mathcal{T}}, \tag{36c}$$

By contrast with (29), we now find that when $\tilde{\mathcal{C}} \sim \delta^{-(n^2+3n+1)/(n+1)^2}$, the stretching rate $\check{\mathcal{U}}_{\check{\mathcal{X}}}$ and therefore the surface slope $\check{\mathcal{H}}_{\check{\mathcal{X}}}$ are no longer negligible, and unlike in (31), we can no longer neglect the driving stress in the force balance of the ice tongue. In fact, when $\tilde{\mathcal{C}} \gg \delta^{-(n^2+3n+1)/(n+1)^2}$, the driving stress dominates lateral shear stress.

Recall that $\tilde{\mathcal{C}} = \delta^{n^2/(n+1)^2}\mathcal{C}$, and that $\check{\mathcal{H}} = \delta^{n^2/(n+1)}\mathcal{H}\tilde{\mathcal{C}}^{n/(n^2+3n+1)} \sim O(1)$. As we require $\mathcal{H} \sim \mathcal{L}$, it follows that $\mathcal{L} \sim \delta^{-n^2/(n+1)}\tilde{\mathcal{C}}^{-n/(n^2+3n+1)}$, and we have from the definitions of $\mathcal{C}$ and $\mathcal{L}$ with $m = 1/n$ that

$$\frac{\Gamma W^{(n+1)/n}}{\Lambda} = \frac{\mathcal{C}}{\mathcal{L}} \sim \tilde{\mathcal{C}}^{(n+1)^2/(n^2+3n+1)} \sim \delta;$$

the parameter regime in which a floating shelf no longer simply buttresses the grounding line is

$$\Gamma \sim \delta^{-(2n^2+3n+1)/(n+1)^2}\Lambda W^{-(n+1)/n}.$$

5.3 Large flotation thickness

A different, simple limit in which analytical formulae can be derived is that of a long ice tongue. This limit occurs when $\mathcal{H}_f \gg 1$, and we do not need to consider $(1-r)$ as small. The appropriate rescaling is

$$\mathcal{H} = \mathcal{H}_f\hat{\mathcal{H}}, \qquad \mathcal{U} = \mathcal{H}_f^{-1}\hat{\mathcal{U}}, \qquad \mathcal{X} = \mathcal{H}_f^{(n+1)/n}\hat{\mathcal{X}}.$$

Then

$$\begin{aligned} 4\mathcal{H}_f^{-(n+1)^2/n^2}(\hat{\mathcal{H}}|\hat{\mathcal{U}}_{\hat{\mathcal{X}}}|^{1/n-1}\hat{\mathcal{U}}_{\hat{\mathcal{X}}})_{\hat{\mathcal{X}}} - \hat{\mathcal{H}}|\hat{\mathcal{U}}|^{1/n-1}\hat{\mathcal{U}} \\ -\theta\mathcal{C}\mathcal{H}_g^{(1-mn-n)/n}|\hat{\mathcal{U}}|^{m-1}\hat{\mathcal{U}} - [1-(1-\theta)r]\,\hat{\mathcal{H}}\hat{\mathcal{H}}_{\hat{\mathcal{X}}} = 0, \end{aligned} \tag{37a}$$

$$(\mathcal{H}\mathcal{U})_{\mathcal{X}} = 0 \tag{37b}$$

where $\theta = 1$, for $\hat{\mathcal{H}} > 1$ and 0 otherwise. In addition, we matching conditions

$$\hat{\mathcal{U}}\hat{\mathcal{H}} \sim 1 \qquad |\hat{\mathcal{U}}|^{1/n-1} \sim \hat{\mathcal{U}}^{-3}\hat{\mathcal{U}}_{\hat{\mathcal{X}}}, \qquad \hat{\mathcal{U}} \to 0. \tag{37c}$$

as $\hat{\mathcal{X}} \to -\infty$ Assume that the grounding line is at $\hat{\mathcal{X}} = 0$, where $\hat{\mathcal{H}} = 1$,

At leading order in $\mathcal{H}_f \gg 1$, (37a) becomes a local force balance between driving stress, basal and lateral drag. We have

$$\hat{\mathcal{H}}|\hat{\mathcal{U}}|^{1/n-1}\hat{\mathcal{U}} - -\theta\mathcal{C}\mathcal{H}_g^{(1-mn-n)/n}|\hat{\mathcal{U}}|^{m-1}\hat{\mathcal{U}} - [1-(1-\theta)r]\,\hat{\mathcal{H}}\hat{\mathcal{H}}_{\hat{\mathcal{X}}} = 0. \tag{38}$$

We focus on the floating portion, where $\theta = 0$, and except near its end at $\hat{\mathcal{X}} = \hat{\mathcal{X}}_c$, we have

$$1 = \hat{\mathcal{H}}\hat{\mathcal{U}} = -(1-r)^n \hat{\mathcal{H}} |\hat{\mathcal{H}}_{\hat{\mathcal{X}}}|^{n-1} \hat{\mathcal{H}}_{\hat{\mathcal{X}}} \tag{39}$$

Key to finding the flux law in the large $\mathcal{H}_f$ limit is now to ensure that this solution also satisfies the appropriate boundary conditions at the glacier terminus. This requires us to construct an additional extensional stress boundary layer to the shallow-ice type problem (38): the original boundary layer of the main text has decomposed into an outer and inner region of its own. Assuming $\mathcal{L}$ is of $O(1)$, the appropriate rescaling is

$$\grave{H} = (1-r)^{n/(n+1)} \mathcal{H}_f \hat{\mathcal{H}}, \qquad \grave{\mathcal{U}} = (1-r)^{-n/(n+1)} \mathcal{H}_f^{-1} \hat{\mathcal{U}}, \qquad \grave{\mathcal{X}} = \hat{\mathcal{H}}_g^{n/(n+1)} (\hat{\mathcal{X}} - \hat{\mathcal{X}}_c)$$

and the boundary layer takes the form

$$4(\grave{\mathcal{H}} |\grave{\mathcal{U}}_{\grave{\mathcal{X}}}|^{1/n-1} \grave{\mathcal{U}}_{\grave{\mathcal{X}}})_{\grave{\mathcal{X}}} - \grave{\mathcal{H}} |\grave{\mathcal{U}}|^{1/n-1} \grave{\mathcal{U}} - \grave{\mathcal{H}} \grave{\mathcal{H}}_{\grave{\mathcal{X}}} = 0 \qquad \text{for } \grave{\mathcal{X}} < 0, \tag{40a}$$

$$(\grave{\mathcal{H}} \grave{\mathcal{U}})_{\grave{\mathcal{X}}} = 0, \qquad \text{for } \grave{\mathcal{X}} < 0, \tag{40b}$$

$$\grave{\mathcal{H}} |\grave{\mathcal{U}}_{\grave{\mathcal{X}}}|^{1/n-1} \grave{\mathcal{U}}_{\grave{\mathcal{X}}} = \grave{\mathcal{H}}^2/2 \qquad \text{at } \grave{\mathcal{X}} = 0, \tag{40c}$$

$$\grave{\mathcal{H}} = 2\mathcal{L}(1-r)^{n/(n+1)} \qquad \text{at } \grave{\mathcal{X}} = 0 \tag{40d}$$

$$\grave{\mathcal{H}} \grave{\mathcal{U}} \to 1, \qquad \grave{\mathcal{U}}|^{1/n-1} \grave{\mathcal{U}} \sim -\grave{\mathcal{U}}^{-3} \grave{\mathcal{U}}_{\grave{\mathcal{X}}}, \qquad \grave{\mathcal{U}} \to 0 \qquad \text{as } \grave{\mathcal{X}} \to -\infty \tag{40e}$$

In this boundary layer problem, $2\mathcal{L}(1-r)^{n/(n+1)}$ appears as the only parameter other than the material constant n. By adapting the reformulation as a dynamical system in the main text, we can show that for given n, this calving front extensional stress boundary layer problem is solvable for only a single choice of that parameter, so that

$$1 = (1-r)^n C(n) (2\mathcal{L})^{n+1},$$

for some $\grave{C}$ that depends only on n, and using the definition of $\mathcal{L}$ gives us

$$Q = (1-r)^n C(n) W (2\Lambda)^{n+1}.$$

We finish our discussion of the long ice tongue case by pointing out that the discontinuity in (38) at $\hat{\mathcal{X}} = 0$, $\hat{\mathcal{H}} = 1$ (where θ goes from 1 to 0) corresponds to a boundary layer of its own; in this boundary layer, the gradient $\hat{\mathcal{U}}_{\hat{\mathcal{X}}}$ however merely undergoes an $O(1)$ change while $\hat{\mathcal{H}}$ and $\hat{\mathcal{U}}$ are unchanged at leading order. The boundary layer is therefore passive with regard to determining the ice flux, but it does become a noticeable feature when looking at solutions of the full boundary value problem near the grounding line, as is the case in the main text.

To describe this additional boundary layer in detail, we can put

$$\check{X} = \mathcal{H}_g^{(n+1)^2/n^2}\hat{\mathcal{X}}, \qquad \check{\mathcal{T}} = |\hat{\mathcal{U}}_{\hat{\mathcal{X}}}|^{1/n-1}\hat{\mathcal{U}}_{\hat{\mathcal{X}}}, \qquad \check{\mathcal{H}} = \mathcal{H}_g^{(n+1)^2/n^2}(\hat{\mathcal{H}} - 1),$$

$$\check{\mathcal{U}} = \mathcal{H}_g^{(n+1)^2/n^2}(\hat{\mathcal{U}} - 1)$$

where $\check{\mathcal{H}}$ and $\check{\mathcal{U}}$ represent small corrections to the zeroth order values of $\hat{\mathcal{H}}$ and $\hat{\mathcal{U}}$ near $\hat{\mathcal{X}} = 0$. Then, at leading order,

$$4\check{\mathcal{T}}_{\check{\mathcal{X}}} - 1 - \theta\mathcal{C}\mathcal{H}_g^{(1-nm-n)/n} - [1 - (1-\theta)r]\check{\mathcal{H}}_{\check{\mathcal{X}}} = 0, \tag{41}$$

$$\check{\mathcal{H}}_{\check{\mathcal{X}}} + \check{\mathcal{U}}_{\check{\mathcal{X}}} = 0, \tag{42}$$

$$\check{\mathcal{U}}_{\check{\mathcal{X}}} = |\check{\mathcal{T}}|^{n-1}\check{\mathcal{T}} \tag{43}$$

with $\theta = 1$ if $\check{\mathcal{H}} \geq 0$, and $\theta = 0$ otherwise. This can be reduced to

$$4\check{\mathcal{T}}_{\check{\mathcal{X}}} - 1 - \theta\mathcal{C}\mathcal{H}_g^{(1-nm-n)/n} + [1 - (1-\theta)r]|\check{\mathcal{T}}|^{n-1}\check{\mathcal{T}}$$

with $\theta = 0$ for $\check{\mathcal{X}} > 0$ and $\theta = 1$ otherwise, combined with matching conditions

$$\check{\mathcal{H}}_{\check{\mathcal{X}}} = -\check{\mathcal{U}}_{\check{\mathcal{X}}} = -|\check{\mathcal{T}}|^{n-1}\check{\mathcal{T}} \sim -\left[1 + \theta\mathcal{C}\mathcal{H}_g^{(1-nm-n)/n}\right]/[1 - (1-\theta)r],$$

which is straightforward to solve (at least in principle; in practice, it would require numerical integration, but the point is that a solution exists). Importantly, for the outer problem, $\hat{\mathcal{H}}$ and $\hat{\mathcal{U}}$ are both continuous across the layer at leading order.

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