

1 **Basal buoyancy and fast moving glaciers: in defense of**
2 **analytic force balance**

3 C.J. van der Veen

4 Department of Geography and Atmospheric Science, University of Kansas, 203 Lindley Hall, 1475
5 Jayhawk Blvd, Lawrence, KS 66045

6

7 **Abstract.** The geometric approach to force balance advocated by T. Hughes in a series of publications has
8 challenged the analytic approach by implying that the latter does not adequately account for basal buoyancy
9 on ice streams, thereby neglecting the contribution to the gravitational driving force associated with this
10 basal buoyancy. Application of the geometric approach to Byrd Glacier, Antarctica, yields physically
11 unrealistic results and it is argued that this is because of a key limiting assumption in the geometric
12 approach. A more traditional analytical treatment of force balance shows that basal buoyancy does not
13 affect the balance of forces on ice streams, except locally perhaps, through bridging effects.

14

15 **1. Introduction**

16 Ice streams are fast-moving rivers of ice embedded in the more sluggish-moving main body of ice sheets,
17 and are responsible for the bulk of drainage from the interior in West Antarctica. Most ice streams start
18 well upstream from the coast, some extending several hundreds of km into the interior, and drain into
19 floating ice shelves or ice tongues and are believed to represent the transition from inland-style “sheet
20 flow” to ice-shelf spreading. The nature of this transition remains under debate, however.

21 In a long series of papers, T. Hughes presents the geometric approach to the balance of forces acting on ice
22 shelves, ice streams, and interior ice (Hughes, 1986, 1992, 1998, 2003, 2009a, 2009b, 2012; Hughes et al.,
23 2011, 2016). Rather than working his way through the basic equations, as done by most other
24 investigators, including Van der Veen and Whillans (1989) and Van der Veen (2013), he presents
25 derivations based on graphical interpretation of triangles representing forces acting on an ice column. In
26 essence, the transition in flow regime is achieved by introducing a basal buoyancy factor that describes the
27 gradual ice-bed decoupling towards the grounding line.

28 The idea of basal buoyancy has been invoked many times before in glaciology, in particular in the context
29 of formulating a sliding relation. In many models, the sliding speed is assumed to be inversely proportional
30 to the “effective basal pressure” defined as the difference between the weight of the overlying ice and the
31 pressure in the subglacial drainage system. Intuitively, this approach may seem to make sense: as the
32 subglacial water pressure increases, the normal force on the bed should be reduced, thus allowing the
33 glacier to move faster. However, this does not affect the balance of forces in the horizontal direction, as
34 suggested by Hughes (2008, 2012).

35 The objective of this brief note is to evaluate the implications of Hughes’ geometric approach to force
36 balance by applying the results to Byrd Glacier, East Antarctica.

37 **2. Force balance: analytical approach**

38 Analytical treatments of glacier force balance are numerous and derivations of the depth-integrated force-
39 balance equations are now standard fare in most glaciology textbooks. In most cases, this balance of forces
40 is discussed in terms of stress deviators, defined as the full stress minus the hydrostatic pressure. This is
41 done because the flow law for glacier ice relates strain rates to stress deviators. That is

42
$$\sigma'_{ij} = \sigma_{ij} - \frac{1}{3}\delta_{ij}[\sigma_{xx} + \sigma_{yy} + \sigma_{zz}] \quad (1)$$

43 where the prime denotes the stress deviator and unprimed stresses are full stresses, and $\delta_{ij} = 1$ for $i = j$ and
44 $\delta_{ij} = 0$ for $i \neq j$. Deviatoric stresses are called for in the flow law for glacier ice because the rate of
45 deformation is in good approximation independent of the hydrostatic pressure. However, the use of
46 deviatoric stresses in discussing the balance of forces unnecessarily complicates the interpretation because

47 the longitudinal deviatoric stress in one direction depends on the full normal stresses in all three directions
 48 of a Cartesian coordinate system. It is more convenient to consider stresses in a glacier as the sum of the
 49 stress due to the weight of the ice (lithostatic stress) and stresses, R_{ij} , due to the flow (resistive stresses).
 50 This partitioning makes a clearer distinction between action and reaction in glacier dynamics (Whillans,
 51 1987) and follows common practice in geophysics (Engelder, 1993, p. 10; Turcotte and Schubert, 2002, p.
 52 77).

53 It may be noted that the term “resistive” stress is an unfortunate choice, perhaps, because these stresses do
 54 not necessarily always offer resistance to flow. For example, gradients in longitudinal stress can act in
 55 cooperation with the driving stress in pulling the ice forward. A more appropriate terminology would
 56 perhaps be *flow stress* or, following geophysical terminology, *tectonic stress*. The R_{ij} represent the
 57 stresses that are associated with glacier deformation, as opposed to the lithostatic stress which describes the
 58 action of gravity. However, the existing terminology appears to have made its way into the glaciological
 59 literature (e.g. Cuffey and Paterson, 2010 section 8.2.2) and a name change at this stage likely would
 60 introduce even more confusion.

61 Van der Veen (2013, sect. 3.1) presents a derivation of the column-average balance equations by
 62 integrating the momentum balance equations over the full ice thickness. Van der Veen and Payne (2004)
 63 and Van der Veen (2013, sect. 3.2) present a discussion of force balance based on geometric arguments
 64 and, not surprisingly, arrive at the same result. Without loss of generality, flow in one horizontal direction
 65 may be considered. That is, the horizontal x-axis is chosen in the direction of flow and it is assumed that
 66 there is no component of flow in the other horizontal y-direction. The z-axis is vertical upward, with $z = 0$
 67 at sea level. Force balance in the flow direction is then described by the following equation (Van der Veen
 68 and Whillans, 1989; Van der Veen, 2013, sect. 3.1):

$$69 \quad \tau_{dx} = \tau_{bx} - \frac{\partial}{\partial x}(H\tilde{R}_{xx}) - \frac{\partial}{\partial y}(H\tilde{R}_{xy}) \quad (2)$$

70 In this expression, τ_{dx} denotes the gravitational driving stress, defined as

$$71 \quad \tau_{dx} = -\rho g H \frac{\partial h}{\partial x} \quad (3)$$

72 where ρ represents the density of ice, g the gravitational acceleration, H the ice thickness, and h the
 73 elevation of the upper ice surface. The terms on the right-hand side of equation (2) represent the resistance
 74 to flow associated with, respectively, drag at the glacier base, gradients in longitudinal stress (“pulling
 75 power”) and lateral drag arising from shear between the faster-moving ice stream and the near-stagnant
 76 interstream ridges or fjord walls. The tilde (\sim) denotes depth-averaged values. Resistive stresses are
 77 defined following Van der Veen and Whillans (1989) as:

$$78 \quad R_{xx} = \sigma_{xx} + \rho g(h - z) \quad (4)$$

79 $R_{xy} = \sigma_{xy}$ (5)

80 where σ_{ij} represents the full stress, and $-\rho g(h - z)$ the lithostatic stress (weight of the ice above) at depth z .
 81 The balance equation (2) is exact. No approximations are involved in deriving this expression from the
 82 basic equations describing the balance of forces on a segment of ice (Van der Veen and Whillans, 1989;
 83 Van der Veen, 2013, sect. 3.1). Consequently, this equation applies to free-floating ice shelves where the
 84 gravitational driving stress is balanced entirely by gradients in longitudinal stress, yielding the classic
 85 Weertman (1957) solution (Van der Veen, 2013, sect. 4.5), as well as laminar flow with basal drag
 86 providing sole resistance to flow (Van der Veen, 2013, sect. 4.2). Except for these two end-member
 87 solutions, equation (2) does not permit analytical solutions without making additional assumptions.
 88 Nevertheless, because no approximations were made in its derivation, balance equation (2) applies equally
 89 well to transitory flow regimes such as ice streams and outlet glaciers.
 90 Integrating the balance equation over the width of the flowband simplifies the resistive term associated with
 91 drag at the lateral margins. Denoting the lateral shear stress at the margins by τ_s (assumed to have the
 92 same magnitude but opposite signs at both lateral margins), and glacier width by W , lateral resistance on a
 93 section of glacier of unit width is (Van der Veen, 2013, eq. (4.39))

94 $F_s = \frac{2H\tau_s}{W}$ (6)

95 and the width-averaged force-balance equation becomes

96 $-\rho g H \frac{\partial h}{\partial x} = \tau_{bx} - \frac{\partial}{\partial x} (H \tilde{R}_{xx}) + \frac{2H\tau_s}{W}$ (7)

97 with the understanding that all terms are averaged over the flowband width (or, equivalently, considered
 98 constant across the flowband), and equation (3) has been substituted for the driving stress on the left-hand
 99 side. Note that contrary to what Hughes (2008, p. 53) states, lateral drag does not vanish at the center of a
 100 glacier. While the shear stress, R_{xy} , is zero at the centerline, its transverse derivative and thereby
 101 resistance from lateral drag, is not zero there. In fact, according to equation (6), this resistance is constant
 102 across the glacier width.

103 The geometric approach developed by Hughes arrives at a similar balance equation, namely

104 $-\rho g H \frac{\Delta h}{\Delta x} = \tau_b - \frac{\Delta H \sigma_F}{\Delta x} + \frac{2H\tau_s}{W}$ (8)

105 (Hughes, 2003, eq. (36)) or, taking the limit $\Delta x \rightarrow 0$

106 $-\rho g H \frac{\partial h}{\partial x} = \tau_b - \frac{\partial H \sigma_F}{\partial x} + \frac{2H\tau_s}{W}$ (9)

107 In these balance equations, σ_F is related to the deviatoric tensile stress; its exact interpretation has evolved
 108 over the years. To avoid unnecessary confusion, a consistent notation is used in the following discussion,

109 based on Hughes (2008, 2012). Comparison of equations (7) and (9) shows that $\sigma_F = \tilde{R}_{xx}$. It is the way
 110 this stress is calculated that sets Hughes' geometric approach apart from the analytical approach. In
 111 essence, this stress is linked to basal buoyancy and, in later versions, downglacier-integrated resistance
 112 from basal and lateral drag. While the force balance equation (7) does not imply any assumption about the
 113 depth-variation in the longitudinal resistive stress, R_{xx} , Hughes (2003) explicitly argues that both σ_F and
 114 the associated stretching rate, $\dot{\epsilon}_{xx}$, must be constant in the vertical direction.

115 3. Force balance: geometric approach

116 Discussing force balance for stream flow, Hughes (2008, section 11) equates σ_F with a basal buoyancy
 117 factor, ϕ , as

$$118 \quad \sigma_F = \frac{\rho_w g H}{2} \phi^2 \quad (10)$$

119 where

$$120 \quad \phi = \frac{\rho_w H_w}{\rho H} = \frac{P_w}{P_i} \quad (11)$$

121 is determined by the ratio of the areal average water pressure under the ice, and basal ice pressure (or
 122 weight of the ice column); ρ_w represents the density of sea water. For a floating ice shelf, $\phi = 1$, and
 123 expression (10) reduces to the solution for a free-floating ice shelf spreading in the x-direction only
 124 (Weertman, 1957; Van der Veen, 2013, sect. 4.5). For inland-style flow, $\phi = 0$, and the lamellar flow
 125 solution can be derived. For ice streams and outlet glaciers that represent the transition from interior-style
 126 flow to ice-shelf spreading, $0 < \phi < 1$. In first-order approximation

$$127 \quad \phi = \frac{H_0}{H(x)} \quad (12)$$

128 where H_0 represents the thickness at the grounding line, and $H(x)$ the ice thickness at some distance x
 129 upstream of the grounding line (Hughes, 2008, eq. (11.11)). This relation is robust and a decrease in ϕ
 130 going upglacier from the grounding line increases ice-bed coupling and generally yields a concave surface
 131 profile (Hughes, 2008, p. 58).

132 Hughes (2008) takes the geometric approach to another level and relates all resistance to flow on ice
 133 streams to the basal buoyancy factor, ϕ . In addition to relating the longitudinal stress deviator to this factor,
 134 lateral and basal drags are linked to ϕ as (Hughes, 2008, table 12.1; see also Hughes, 2009a,b; Hughes,
 135 2012, table 12.1; Hughes et al., 2016, eqs. (12) – (17))

$$136 \quad \tau_b = -\rho_w g H (1 - \phi)^2 \frac{\partial h}{\partial x} - \rho_w g H^2 (1 - \phi) \frac{\partial \phi}{\partial x} \quad (13)$$

137
$$F_s = \frac{2H\tau_s}{W} = -2\rho g H\phi(1-\phi)\frac{\partial h}{\partial x} - \frac{1}{2}\rho g HW(1-2\phi)\frac{\partial \phi}{\partial x} \quad (14)$$

138 while the longitudinal stress gradient term is given by

139
$$\frac{\partial H\sigma_F}{\partial x} = \rho g H\phi\left(\phi\frac{\partial h}{\partial x} + H\frac{\partial \phi}{\partial x}\right) \quad (15)$$

140 These equations are derived without consideration of ice velocity or physical properties of the ice
 141 (temperature, stiffness, fabric development, etc.), or, for that matter, basal water availability and balance.
 142 Presumably, all these factors are somehow reflected in the ice-stream geometry and the inferred basal
 143 buoyancy.

144 **4. Geometric approach: application to Byrd Glacier, Antarctica**

145 Balance of forces on Byrd Glacier, East Antarctica, was first discussed by Whillans et al. (1989) who used
 146 measurements of surface velocity and surface topography derived from repeat aerial photogrammetry, to
 147 evaluate the relative roles of lateral drag, gradients in longitudinal stress, and basal drag in resisting the
 148 gravitational driving stress. Van der Veen et al. (2014) reconsidered these calculations and also
 149 investigated the effect of drainage of two sub-glacial lakes in the catchment region. Both studies employed
 150 the analytical force-balance approach.

151 Reusch and Hughes (2003), Hughes (2009a), Hughes et al. (2011), and Hughes et al. (2016) discuss force
 152 balance on Byrd Glacier from the geometrical perspective and take issue with the analytical approach of
 153 Whillans et al. (1989). None of these studies explicitly shows how the various resistive forces vary along
 154 the glacier and, instead, largely base their discussion on how the basal buoyancy, ϕ , varies upstream of the
 155 grounding line. Therefore, to fully appreciate the implications of the geometrical approach, equations (13)
 156 – (15) are applied here to evaluate all terms in the balance of forces.

157 The geometry is shown in Figure 1 (Van der Veen et al., 2014, fig. 6). Only the lower 30 km stretch
 158 upstream of the grounding line (at $x = -10$ km) is considered here because that is the region laterally
 159 bounded by near-parallel ford walls. Also shown in Figure 1 is the basal buoyancy factor calculated from
 160 eq. (12); ϕ increases from around 0.7 a little more than 30 km upstream of the grounding line, to 1 where
 161 the ice starts to float. While there is nothing in particular wrong or disturbing about this basal buoyancy
 162 factor, the situation becomes more problematic when the actual forces are considered.

163 The average driving stress is ~160 kPa, but shows large spatial variations that appear to be temporally fixed
 164 (Figure 2). Gradients in longitudinal stress are mostly negative, averaging -140 kPa along the flowline,
 165 implying that, except in a few isolated locations, this term acts in the same directions as the driving stress,
 166 draining the grounded ice into the Ross Ice Shelf. To maintain balance of forces, flow resistance is
 167 partitioned between basal drag (~53 kPa) and lateral drag (~247 kPa). In the geometric approach, the bulk
 168 of flow resistance is associated with lateral drag and basal drag supports only about 1/3 of the driving
 169 stress. The finding that longitudinal stress gradients act in cooperation with the driving stress over a

170 distance of more than 30 km is surprising and there is no credible physical mechanism that can explain this.
 171 Even on a free-floating ice shelf, where other sources of flow resistance may be neglected, gradients in
 172 longitudinal stress arising from water pressure act to oppose the driving stress (Weertman, 1957; Van der
 173 Veen, 2013, sect. 4.5). Hughes et al. (2016, p. 201) argue that the water buttressing produces a backstress
 174 in the longitudinal force balance, and that this is a real stress that is obscured using continuum mechanics in
 175 the conventional analytical approach. According to Hughes (2008, 2012), this stress, or “pulling power”
 176 results in the overestimation of longitudinal stress gradients, adding to the driving stress.

177 **5. Limitation of the geometric approach**

178 To understand the limitation in the geometrical approach to force balance, consider the forces along an ice
 179 stream flow line as discussed in Hughes (2008, p. 53 ff.) (see also figure 1 in Hughes (2003), and Hughes
 180 (2012, section 11)). The geometry is shown in Figure 3. While Hughes (2008, p. 53; 2012, p. 66)
 181 erroneously states that resistance from lateral drag vanishes at the centerline of an ice stream and therefore
 182 does not include this source of resistance in his discussion, this has no significant impact on the following
 183 discussion – lateral drag can be readily added to the basal drag term without altering the general tenets of
 184 the analysis.

185 According to Hughes (2008, 2012), the gravitational driving force at x is

$$186 \quad F_g = \text{area ADF} = \frac{1}{2} \rho g H^2 \quad (16)$$

187 and this force must be balanced by longitudinal resisting forces consisting of a “water buttressing force”
 188 (area CDE), a tensile force (area BCE), and a basal drag force (area ABEF). The basal drag force equals
 189 integrated basal resistance from the grounding line to the upglacier location (integrated resistance from
 190 lateral drag could also be included in this term). The area of each triangle is obtained from the familiar
 191 formula (base \times height) / 2, where the base either equals the ice overburden pressure ($DF = \rho g H$) or water
 192 pressure ($DE = \rho_w g H$), and the height equals the ice thickness ($AD = H$), flotation height ($BD =$
 193 $H_f = (\rho_w / \rho) H_w$), or the piezometric height ($CD = H_w = P_w / (\rho_w g)$). Thus, each of the resistive
 194 terms can be evaluated as a function of local ice thickness and water pressure. The reason why, for
 195 example, area ABEF should be associated with basal drag force (or basal plus lateral drag), remains unclear
 196 but is irrelevant.

197 The problem with this reasoning is that F_g *does not* represent the gravitational driving force. Rather, this
 198 force equals the lithostatic force associated with the weight of ice. When considering horizontal forces at
 199 any location, this force is balanced exactly by an equal but opposite force from ice of equal thickness on the
 200 left of the vertical line AD, except at the calving front. In other words, adhering to the geometric
 201 representation, triangle ADF is balanced by the mirror triangle ADP (Figure 4a), whether one considers an
 202 ice shelf, ice stream, or interior ice. The gravitational force that drives glacier flow is associated with
 203 gradients in lithostatic stress (Figure 4b). A correct geometry-based discussion of force balance would

204 consider the difference between lithostatic stress at x and at some location $x + \Delta x$ downglacier, and, in the
 205 case of a sloping bed, lithostatic stress acting on the bed, and the difference between longitudinal stress at
 206 both locations, in addition to basal and lateral drag acting over the distance considered. Doing so gives the
 207 balance equation (7) with the term on the left-hand side representing the driving stress (Van der Veen and
 208 Payne, 2004; Van der Veen, 2013, section 3.2).

209 It is not possible to relate resistive forces at any location to point values such as basal water pressure or
 210 weight of the ice at location x . While resistive stresses, such as R_{xx} , can be evaluated at specific points,
 211 resistance to flow is associated with gradients in these stresses (see, e.g., Van der Veen, 2013, figure 3.1
 212 and eqs. (3.8) – (3.9)). Balance of forces is only meaningful if applied to flowline segments, not single
 213 locations. Consequently, the concept of force balance at any location is inherently flawed. While many, if
 214 not most, glaciologists, Van der Veen (2013) included, often refer to driving stress or basal drag at location
 215 x , it would be more appropriate to refer to these quantities as areal averages. If the surface slope is
 216 calculated over a distance $2\Delta x$, the associated driving stress is the average over the interval $(x - \Delta x, x +$
 217 $\Delta x)$, and similarly for basal drag. Nuancing common parlance to reflect this subtlety would render many
 218 discussions of glacier dynamics unnecessarily cumbersome and should be superfluous for most readers
 219 understanding the fundamentals of glacier dynamics.

220 6. Discussion

221 While the geometric force balance approach is severely limited, it is worth exploring the central premise of
 222 Hughes' ideas, namely that the transition from sheet flow to shelf flow is achieved through basal buoyancy,
 223 with interior ice firmly grounded on bedrock and ice shelves floating in sea water. It should be noted that
 224 for both these end member solutions, at any location the weight of an ice column is fully supported from
 225 directly below: terra firma in the case of grounded ice, and sea water for ice shelves.

226 While not immediately obvious, the role of varying subglacial water pressure is included in the force-
 227 balance equation (2), namely through bridging effects (Van der Veen, 2013, sect. 3.4). To clarify this,
 228 consider that resistive stresses are linked to strain rates, or velocity gradients, by invoking Glen's flow law
 229 for glacier ice (Van der Veen and Whillans, 1989; Van der Veen, 2013, sect. 3.3):

$$230 \quad R_{xx} = B\dot{\epsilon}_e^{1/n-1} (2\dot{\epsilon}_{xx} + \dot{\epsilon}_{yy}) + R_{zz} \quad (17)$$

$$231 \quad R_{xy} = B\dot{\epsilon}_e^{1/n-1} \dot{\epsilon}_{xy} \quad (18)$$

232 Here, B represents the temperature-dependent rate factor, and $n = 3$ the flow-law exponent; $\dot{\epsilon}_e$ is the
 233 effective strain rate defined as the second invariant of the strain-rate tensor. The last term on the right-hand
 234 side of equation (17) is the vertical resistive stress defined as

$$235 \quad R_{zz}(z) = \sigma_{zz} + \rho g(h - z) \quad (19)$$

236 The stress R_{zz} represents the difference between the full vertical stress, σ_{zz} , and the lithostatic stress, or
 237 weight of the ice above some level. This term arises in equation (7) because Glenn's flow law relates strain
 238 rates to deviatoric stresses, rather than full stresses.

239 For brevity of notation, the along-flow resistive stress is written as the sum of a contribution associated
 240 with along-flow gradients in velocity (first term on the right-hand side of equation (17)) and the vertical
 241 resistive stress:

$$242 \quad R_{xx} = R_{xx}^{(0)} + R_{zz} \quad (20)$$

243 Force-balance in the horizontal direction can then also be written as

$$244 \quad \tau_{dx} = \tau_{bx} - \frac{\partial}{\partial x} (H \tilde{R}_{xx}^{(0)}) - \frac{\partial}{\partial y} (H \tilde{R}_{xy}) - \frac{\partial}{\partial x} \int_{h-H}^h R_{zz}(z) dz \quad (21)$$

245 Where the weight of the ice is fully supported by the substrate below, the vertical resistive stress is zero.
 246 This is the assumption usually made when considering the budget of forces acting on glaciers (e.g. Van der
 247 Veen and Whillans, 1989). Locally, however, bridging effects may be important, for example where a
 248 water-filled cavity exists at the ice-bed interface (Van der Veen, 2013, sect. 7.2). Where cavitation occurs
 249 and basal ice becomes separated from the bed, the cavity cannot support the weight of the ice leading to
 250 shear-stress gradients that effectively transfer the weight to surrounding areas where the ice is in contact
 251 with the bed, such that the areal average of the vertical resistive stress is zero. Thus, on a large scale, such
 252 as the length of ice streams and outlet glaciers, basal buoyancy is a non-issue where horizontal force
 253 balance is concerned. Indeed, Hughes (1998, eq. (3.5)) does not include bridging effects in his discussions
 254 and equates the total vertical stress at depth to the lithostatic stress.

255 Basal buoyancy may be important on ice streams and outlet glaciers according to the commonly-adopted
 256 sliding relation in which sliding speed is inversely proportional to the effective basal pressure. Pfeffer
 257 (2007) suggests that this proportionality may explain rapid velocity increases on tidewater glaciers and
 258 Greenland outlet glaciers: as these glaciers thinned and thickness approached flotation, the effective basal
 259 pressure approached zero, resulting in a large increase in sliding velocity. Another possibility is that
 260 increased basal buoyancy reduces basal drag, thereby allowing glaciers to move faster. The importance of
 261 these effects can be evaluated from analysis of time series of surface speed and glacier geometry, or using
 262 numerical models based on the balance equation (7).

263 The primary difference between shelf flow and stream flow is not that on ice shelves the ice weight is
 264 supported by water and on grounded interior ice this weight is supported by the bed below. The main
 265 difference is that, because ice shelves float in water, basal drag is zero and resistance to flow must be
 266 partitioned between gradients in longitudinal stress and lateral drag, whereas for sheet flow, basal drag
 267 provides most resistance to flow. Thus, it would seem reasonable to propose that the transition from sheet
 268 to shelf flow involves a gradual reduction in basal resistance, perhaps associated with the presence within
 269 deforming sediments, or gradual drowning of bed obstacles. As basal drag becomes less important,

270 longitudinal stress gradients and lateral drag must increase and provide most or all resistance to the flow of
271 ice streams.

272 **7. Concluding remarks**

273 The geometrical approach to ice sheet modeling links ice-bed coupling directly to the stresses that resist
274 horizontal gravitational motion (Hughes, 2008, p. 34). This basal buoyancy supposedly translates into a
275 major component of gravitational forcing by which ice sheets discharge ice into the sea (Hughes, 2003).
276 As shown in this contribution, the geometric force balance as presented by Hughes in a series of
277 publications cannot be successfully applied to ice streams and outlet glaciers. This is not to say that a
278 geometric approach is inherently flawed – if implemented correctly it should produce consistent and correct
279 results but this has yet to be achieved.

280 The charge that the analytical force-budget approach fails to account for basal buoyancy and excludes a
281 “water buttressing force” on ice streams is incorrect. Equation (7) describing the depth-integrated balance
282 of horizontal forces is derived without making any simplifying assumptions and applies equally well to
283 floating ice shelves and firmly grounded interior ice. If some force is missing from this equation, this force
284 must also be missing from the momentum balance equations that form the starting point for deriving
285 equation (7).

286 Hughes is correct that ice streams and outlet glaciers represent the transition from sheet flow and shelf flow
287 and that much remains to be understood about the nature of this transition. Advantageously, ongoing rapid
288 changes on many of the outlet glaciers have been well documented through time series of surface elevation
289 and surface velocity. The latter, in particular, are powerful indicators of the distribution of stresses on
290 glaciers because strain rates (velocity gradients) are directly linked to stresses through the flow law for
291 glacier ice. Improved understanding of the dynamics of rapidly-changing ice-sheet components will come
292 from interpretation of strain rates and temporal changes therein.

293 **8. Acknowledgements**

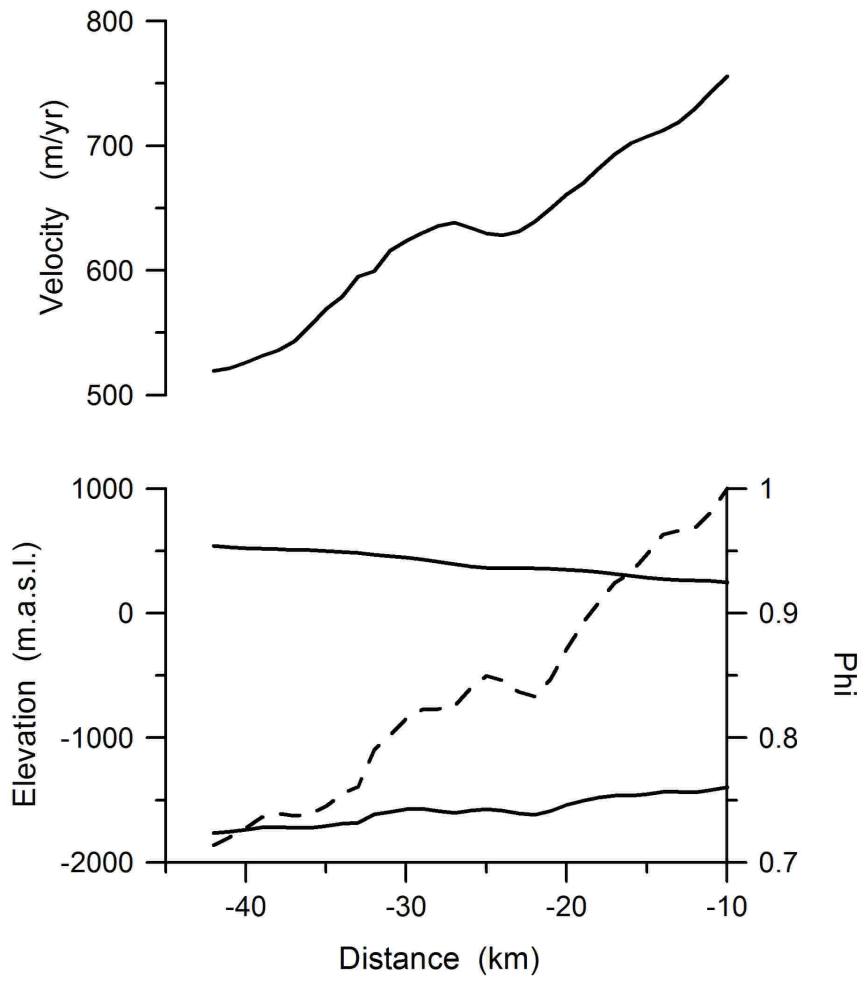
294 I am indebted to Ken Jezek for his continued support and careful reading of this note, and to Leigh Stearns
295 for additional comments. This research was supported NASA grant no. FED0066542 and UNI0072622.

296

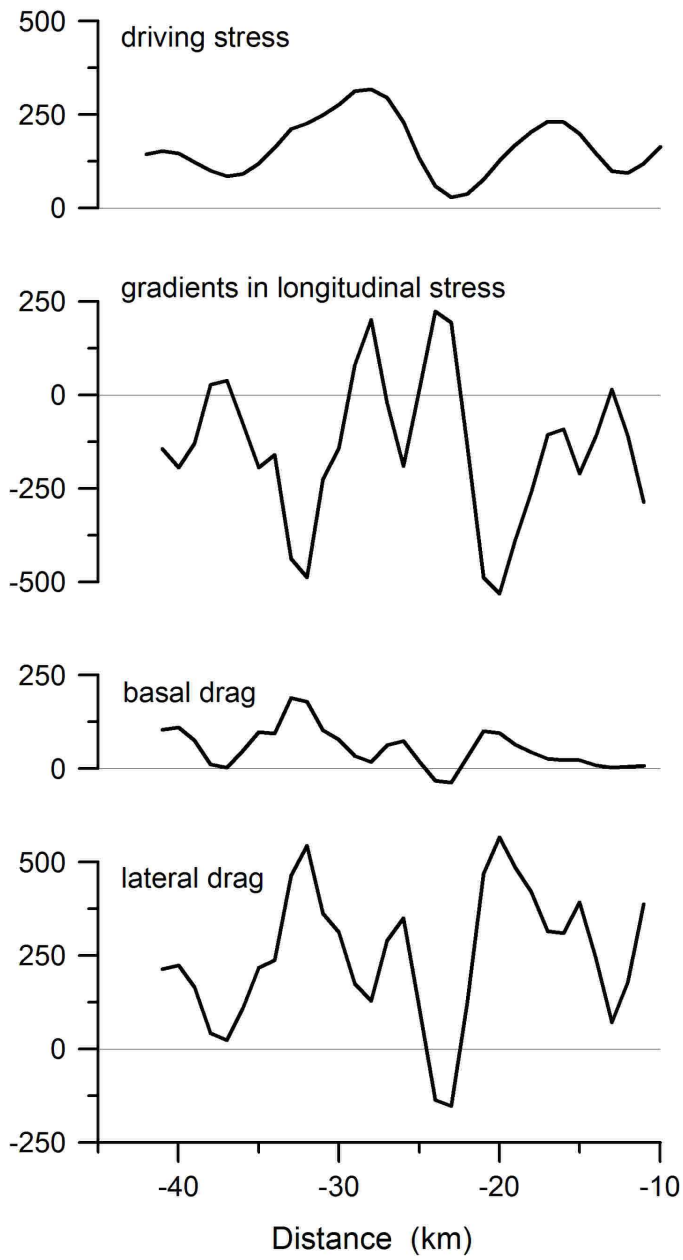
297 **References**

- 298 Cuffey, K.M., and Paterson, W.S.B.: *The Physics of Glaciers (Fourth Edition)*. Burlington MA:
299 Butterworth-Heinemann, 693 pp., 2010.
- 300 Engelder T: *Stress regimes in the lithosphere*. Princeton NJ: Princeton University Press, 457 pp., 1993
- 301 Hughes, T.: The Jakobshavn Effect. *Geophys Res Lett* 13, 46-48, 1986.
- 302 Hughes, T.: On the pulling power of ice streams. *J Glaciol* 38, 125-151, 1992.
- 303 Hughes, T.J.: *Ice Sheets*. New York NY: Oxford University Press, 343 pp., 1998
- 304 Hughes, T.J.: Geometrical force balance in glaciology. *J Geophys Res* 108, 2526,
305 doi:10.1029/2003JB002557, 2003.
- 306 Hughes, T.J. (2008), *Holistic ice sheet modeling: a first-order approach*. University of Maine.
- 307 Hughes, T.: Variations of ice bed coupling beneath and beyond ice streams: The force balance. *J Geophys*
308 *Res*, 114, B01410, doi:10.1029/2008JB005714, 2009a.
- 309 Hughes, T.: Correction to “Variations of ice bed coupling beneath and beyond ice streams: The force
310 balance,” *J Geophys Res* 114, B04499, doi:10.1029/2009JB006426, 2009b.
- 311 Hughes, T.: *Holistic Ice Sheet Modeling. A First-Order Approach*. New York: Nova Publishers, 261 pp.,
312 2012
- 313 Hughes, T., Sargent, A., and Fastook, J.: Ice-bed coupling beneath and beyond ice streams: Byrd Glacier,
314 Antarctica. *J Geophys Res* 116, F03005, doi:10.1029/2010JF001896, 2011.
- 315 Hughes, T., Sargent, A., Fastook, J., Purdon, K., Li, J., Yan, J.-B., and Gogineni, S.: Sheet, stream, and
316 shelf flow as progressive ice-bed uncoupling: Byrd Glacier, Antarctica and Jakobshavn Isbræ,
317 Greenland. *The Cryosphere* 10, 193-225, 2016.
- 318 Pfeffer, W.T.: A simple mechanism for irreversible tidewater glacier retreat. *J Geophys Res* 112, F03S25,
319 doi:10.1029/2006JF000590, 2007.
- 320 Reusch, D., and Hughes, T.: Surface “waves” on Byrd Glacier, Antarctica. *Antarct Sci* 15, 547-555, 2003.
- 321 Turcotte, D L and Schubert, G.: *Geodynamics (Second Edition)*. Cambridge: Cambridge University Press,
322 456 pp., 2002.
- 323 Van der Veen, C.J.: *Fundamentals of Glacier Dynamics (Second Edition)*. Boca Raton: Taylor & Francis,
324 389 pp., 2013.
- 325 Van der Veen, C.J., and Whillans, I.M.: Force budget: 1. Theory and numerical methods. *J Glaciol* 35, 53-
326 60, 1989.

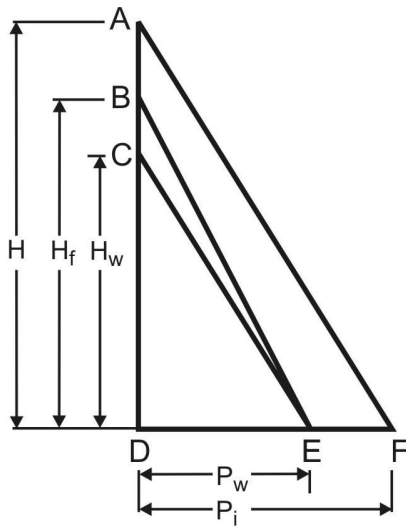
- 327 Van der Veen, C.J., and Payne, A.J.: Modelling land-ice dynamics. In: Mass balance of the cryosphere:
328 observations and modelling of contemporary and future changes (Bamber, J.L., and A.J. Payne, A.J.,
329 eds.). Cambridge: Cambridge University Press, 169-225, 2004.
- 330 Van der Veen, C.J., Stearns, L.A., Johnson, J., and Csatho, B.: Flow dynamics of Byrd Glacier, East
331 Antarctica. *J Glaciol* 60, 1053-1064, 2014.
- 332 Weertman, J.: Deformation of floating ice shelves. *J Glaciol* 3, 38-42, 1957.
- 333 Whillans, I.M.: Force budget of ice sheets. In: Dynamics of the West Antarctic Ice Sheets (Van der Veen,
334 C.J. and Oerlemans, J., eds.). Dordrecht: Reidel, 17-36, 1987.
- 335 Whillans, I.M., Chen, Y.H., Van der Veen, C.J., and Hughes, T.J.: Force budget: III. Application to three-
336 dimensional flow of Byrd Glacier, Antarctica. *J Glaciol* 119, 68-80, 1989.



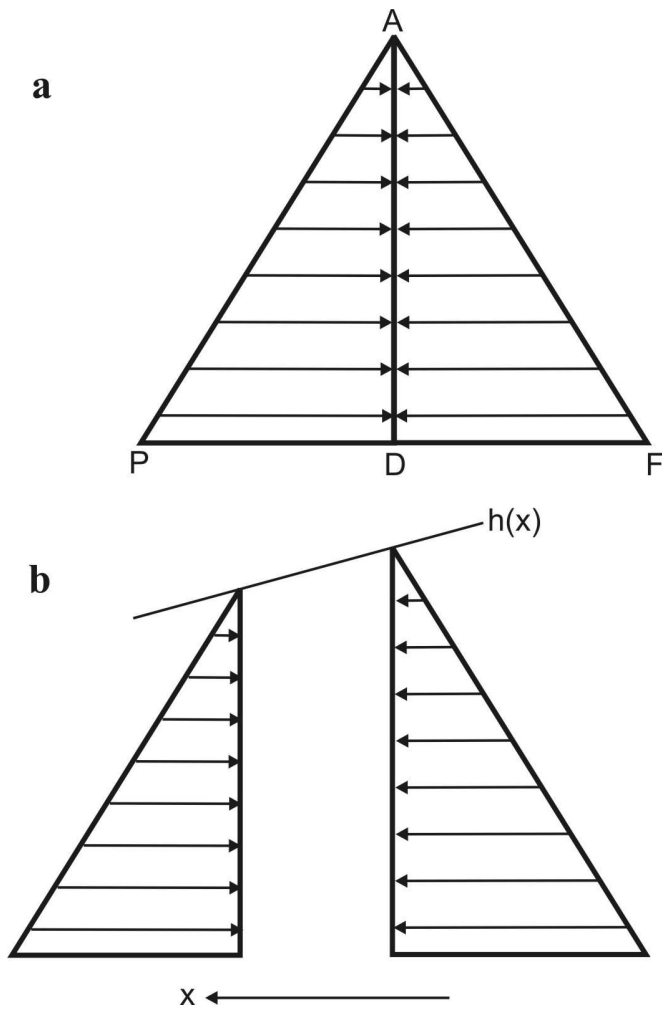
337 Figure 1. Geometry of the lower part of Byrd Glacier, East Antarctica. The dashed line in the lower panel
 338 shows the buoyancy factor, calculated from eq. (12).



339 Figure 2. Force-balance terms on Byrd Glacier according to geometric force balance, eqs. (13) – (14).



340 Figure 3. Geometric force balance according to Hughes (2008). H represents ice thickness, H_f the
 341 flotation height or height of the ice column supported by basal water pressure, and H_w the piezometric
 342 height; P_w and P_i represent the basal water pressure and weight of the ice column, respectively. Ice flow is
 343 from right to left.



344 Figure 4. (a) at any location the lithostatic stress increases linearly with depth from zero at the ice surface
 345 to ρgH at the base; the lithostatic stress from ice on the right of the vertical line AD is balanced by an equal
 346 but opposite lithostatic stress from ice on the right and the area of triangle ADF equals that of triangle
 347 ADP. (b) gradients in lithostatic stress are associated with a sloping ice surface, $h(x)$, resulting in a smaller
 348 lithostatic stress in the downslope direction; the difference between the areas of both triangles is a measure
 349 of the gravitational driving stress responsible for glacier flow.