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Dear Stephen Conford,

Please find hereafter the answers to your comments, concerns and remarks regarding the article entitled :

"Of the gradient accuracy in Full-Stokes ice flow model: basal slipperiness inference"

by Nathan Martin and Jérôme Monnier

Sincerely yours,

N. Martin and J. Monnier

You expressed an important concern about the level of english in the paper, as the other referees did. In order to address this concern, a thorough attention to the correct and sensible use of english will be paid in the revised version of the paper. However, although the following comments have been carefully written, they might contain english mistakes and we apologize in advance.

1 Specific comments

1.1 Gradient and scalar product tests

- Since the concept of directional derivative is of importance throughout the paper, it seems more explicit to write the finite-difference formula as you propose, thus making clearer the distinct meaning of the quantities α (the magnitude that vanishes) and $\delta\boldsymbol{k}$ (the direction). Also the way the Taylor remainder is written is maybe a little specific for the purpose and not needed. The whole equation (16) could then be replaced by:

$$\frac{j(\mathbf{k}+\alpha\delta\mathbf{k})-j(\mathbf{k}-\alpha\delta\mathbf{k})}{2\alpha}=\frac{\partial j}{\partial\mathbf{k}}\cdot\delta\mathbf{k}+O\left(\alpha^2\delta\mathbf{k}^3\right)$$

- The fact that the quantity $|I_\alpha-1|$ remains around 1 is indeed a predictable behavior. Based on a comment made by Referee #1, the continuous adjoint system could be added in this section to show explicitly the terms dropped by the "self-adjoint" approximation and therefore, based on these equations, we propose to add a comment about the value of the quantity $|I_\alpha-1|$ close to 1 for the "self-adjoint" approximation as follows (the following paragraphs in italic has been paste from our answer to Refereee # 1):

Omitting the lateral boundaries, the adjoint system writes (see e.g. Petra et al., "A Newton method for inversion in a nonlinear Stokes ice sheet model "):

$$-\boldsymbol{div}(\underline{\boldsymbol{\Sigma}})=0 \textit{ in } \Omega \quad (1)$$
$$\boldsymbol{div}(\mathbf{v})=0 \textit{ in } \Omega \quad (2)$$
$$\underline{\boldsymbol{\Sigma}}\boldsymbol{n}=\mathbf{u}_s^{obs}-\mathbf{u} \textit{ on } \Gamma_s \quad (3)$$
$$\boldsymbol{\Sigma}_{nt}=\beta^{1/m}\left(|\mathbf{u}_\tau|^{\frac{1-m}{m}}\mathbf{v}_\tau+(m-1)|\mathbf{u}_\tau|^{\frac{1-3m}{m}}(\mathbf{u}_\tau\otimes\mathbf{u}_\tau)\mathbf{v}_\tau\right) \textit{ on } \Gamma_{fr} \quad (4)$$
$$\mathbf{v}\cdot\boldsymbol{n}=0 \textit{ on } \Gamma_{fr} \quad (5)$$

where $\mathbf{v}$ denotes the adjoint velocity. The adjoint stress tensor $\boldsymbol{\Sigma}$ is defined by:

$$\underline{\boldsymbol{\Sigma}}=2\eta(\mathbf{u},n)\left(I+\frac{1-n}{n}\frac{\underline{\boldsymbol{D}}(\mathbf{u})\otimes\underline{\boldsymbol{D}}(\mathbf{u})}{\|\underline{\boldsymbol{D}}(\mathbf{u})\|_F^2}\right)\underline{\boldsymbol{D}}(\mathbf{v})-\boldsymbol{Id}\,q \quad (6)$$

with q denoting the adjoint pressure, I the fourth-order identity tensor applied to order two tensors, $\boldsymbol{Id}$ the second order identity tensor and $'\otimes'$ the tensor product.

This problem is a linear problem in $\mathbf{v}$ and depends on the forward velocity $\mathbf{u}$.

The self-adjoint method consists in neglecting the non-linearity that it to say the dependence of the viscosity and, in the present situation the friction condition, on the solution $\mathbf{u}$. Equivalently, it corresponds to set $n=m=1$ in the adjoint system (1)-(5). It is straightforward from the previous system to see that the corresponding adjoint operator, under this approximation (i.e. with $m=n=1$), is in fact the forward Stokes operator for a Newtonian fluid and a linear friction, hence the "self-adjoint" terminology. This terminology is incorrect since, the forward velocity field $\mathbf{u}$ is generally computed from the non-linear full-Stokes solve and leads then to a velocity-independent yet spatially variable viscosity field and consequently to a non-symmetric problem (which cannot *then represent a self-adjoint operator).*

First, the non-linearity of the forward problem appears in the definition of the adjoint stress given in equation (6)*. The norm of the term* $\dfrac{\underline{\boldsymbol{D}}(\mathbf{u}) \otimes \underline{\boldsymbol{D}}(\mathbf{u})}{\|\underline{\boldsymbol{D}}(\mathbf{u})\|_F^2}$ *is simply one (since* $\|\underline{\boldsymbol{D}} \otimes \underline{\boldsymbol{D}}\| = \|\underline{\boldsymbol{D}}\|_F \times \|\underline{\boldsymbol{D}}\|_F$ *given a choice of the fourth-order tensor norm consistent with the Frobenius matrix norm). And the norm of the identity tensor is known to be greater or equal to one (and typically equal to one for the sup norm). It follows that the terms that are being dropped are comparable to the one that are kept in the "self-adjoint" approximation for* $\dfrac{1-n}{n}$ *close to one (*$2/3$ *for* $n = 3$*). It logically follows that the greater the non-linearity (the value of* n*), the greater the non-linear contribution.*

The other non-linearity comes from the non-linear friction law and appears in equation (4)*. A similar calculation leasd to the exact same conclusion and for* $m > 1$*, the norm of the two terms following the friction coefficient* β *are comparable.*

As pointed out by Reviewer #2 Stephen Cornford, this fact is well observed in the gradient test results performed for the "self-adjoint" approximation which provides a term $|I_\alpha - 1|$ *always around* 1.

As you suggested, these observations could really be of interest and support the results of our work so they could be included in the revised version.

- The minimization problem is solved using a line search algorithm. Therefore, only the direction matters to compute the search direction for the algorithm. The optimal step α along this direction is obtained by minimizing the value of the cost function for this direction with respect to α. So the value of α will be adjusted according to the magnitude of the search direction: it is the linear search step. Finally the magnitude of the computed gradient does not matter and only its direction is important for the optimization routine.

- Regarding the scalar product test mentioned in section 2.5.2, it was mentioned along with the gradient test as part of the *2.5 Validation of the adjoint model* since it is the other standard test performed to validate an implementation for the adjoint solve. No further use of this test is made in the present paper so our suggestion would be to simply remove it from subsection 2.5 and rename the subsection *The gradient test*.

1.2 Discussion of the slip ratio

It is possibly a bit misleading but the precision of the identification process is assessed in terms of relative errors and the absolute error does not have any impact on the result for these test cases. However, it could be interesting to highlight this aspect since it does make a difference in terms of order of magnitude of the error and it can make a difference for real data ; an increasing slip ratio does lead to a smaller amplitude on the inferred β. A comment could be added about this aspect.

2 Presentation issues

- It will be modified in the revised paper

- The sentence is probably too convoluted and should be reformulated but it is not suppose to mean that surface velocity observations does not appear in the momentum equation whereas the friction coefficient does. The role of this sentence was to point out that a major component for ice flows is the state of the bed (that is to say the bedrock itself, the subglacial till, the subglacial hydrology,...) as opposed to the surface which is what we observe, hence the sentence that follows in the paragraph: "This raises questions about, on the one hand, the ability for the surface data to encompass basal conditions and, on the other hand, the potential for inverse methods to recover information".

- The sentence is not correct and should be reformulated. There is no implication between the fact that the cost function becomes more regular thanks to the regularization term and the fact that it introduces a bias toward smoothly varying β. The restriction to smooth β is due to the chosen regularization operator (the gradient). In addition, the fact that the cost function would be more convex is not necessarily true when using this regularization term. It would have been true if the regularization operator was the identity. So in the present case the sentence, built on you proposition, could be for example:
  "This term regularizes the functional to be minimized and introduces a bias toward smoothly varying fields".

3 Technical corrections

- It will be corrected in the revised paper
- It will be modified in the revised version
- It requires to modify many figures but it can be done if necessary.