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Dear Referee #1,

Please find hereafter the answers to your comments, concerns and remarks regarding the article entitled :

"Of the gradient accuracy in Full-Stokes ice flow model: basal slipperiness inference"

by Nathan Martin and Jérôme Monnier

Sincerely yours,

N. Martin and J. Monnier

You expressed an important concern about the fact that our conclusions and results strongly rely on a black-box type adjoint derivation, thus influencing the whole discussion and making the study not applicable to a more generic setup.

Although it is true that the considerations regarding the computational burden when using an automatically derived adjoint code are specific to this situation, all the assessments made regarding the accuracy of the exact adjoint and the "self-adjoint" approximation are generic and valid for any implementation of the adjoint problem (from a discretization of the continuous adjoint equations to a source-to-source differentiation of the forward code). In addition, according to the published litterature, very few research teams have access to an implementation of an exact adjoint solution for the full Stokes problem (whatever the method considered for the implementation) and our results on the accuracy of this method (and the comparison with the accuracy of the "self-adjoint" method commonly used in the glaciology community) are provided here for the first time.

However, as you pointed it out several times, the considerations on the computational cost of the present approach are, indeed, applicable only to this type of approach. But, we believe that automatic differentiation is a recent but spreading tool for the implementation of inverse methods and that the community could be interested in this specific aspect of the inverse approach. For instance, a very similar accumulation method is used in *Goldberg and Heimbach - Parameter and state estimation with a time-dependent adjoint marine ice sheet model, TCD 2013* for computing the adjoint associated to the non-linear L1L2 problem. As a matter of fact, this reference should be added as a glaciological reference using algorithmic differentiation.

It is clearly of interest to provide the continuous adjoint equations when possible and, for the flow considered in the present work, the strong form of the adjoint equations and the effects of the "self-adjoint" approximation are detailed in Section 3(c) of this letter as you suggested. However, the continuous adjoint equations do not allow in general, to accurately assess the loss of precision and would provide a more visual presentation of the approximation yet harder to numerically quantify.

Please find hereafter an answer point-by-point to your comments:

1 Title

(a) We thought of the "of" as a more litterary form (as in "Of mice and men") but maybe "On" is more suited.

(b) the capital "F" was just to somewhat recall the common acronym FS (for Full Stokes) used in the glaciological community to designate the Stokes equation with a power-law rheology. The hyphenation can be corrected.

(c) It is maybe a little confusing. However the gradient provided by a Newton method corresponds to the linear tangent model which provide the same gradient as the adjoint code but restricted to a given direction of derivation so it is not completely misleading. But the title can be modified for a more explicit version.

2 Abstract

(a) Since the parameter β is preceeding the velocity it should be made homogeneous using the term "friction coefficient", as you suggest.

(b) Attention will be paid to consistent use of terminologies in the revised version of the paper. Regarding the differences between the three methods, Section 3 is mostly dedicated to their descriptions but maybe some improvement could be made to refer precisely to each terminology using consistent denominations throughout the paper. It is a rather complicated matter to present in an abstract so we thought it was fare to assume that the reader know about the self-adjoint approximation used in glaciology and why it is not an exact adjoint method. The incomplete adjoint approach cannot really be introduced without several things in mind such as the reverse accumulation technique and it seems difficult for the abstract and the introduction to provide the exact meaning of this method but only state that it lies inbetween the two others as a tunable one and address the reader to section 3 for the details.

(c) The accuracy of the adjoint-based gradient is studied itself mainly in section 3 using the gradient test (which is a classical tool to assess the quality of the gradient provided by the adjoint code) and the accuracy of the "self-adjoint "gradient is compared using the same test. Sections 4 and 5 present a lot of variational (*i.e.* adjoint-based) data assimilation runs to infer the friction coefficient in various context of frequency, density and noise. The results of identifications are compared in terms of precision using Morozov's discrepancy principe and the computed gradients are also plotted for various slip ratio. There is different ways of solving an inverse problem and the two most important would be the stochastic approach (filter-based) and the variational approach (adjoint-based). It seems important to precise which one is considered here and "inverse problems"seems a little bit to general.

(d) Since there are few statements in the litterature about what level of information can be retrieved about the state of the bedrock from surface velocity observations through parameter identification, it makes sense that Petra et al. compared their results to those of Gudmundsson et al. at least to demonstrate that the exact adjoint method leads, as one can expect, to a smaller lower bound on reachable wavelengths in the reconstruction of the basal friction coefficient. Concerning the comparison of our results with those of Petra et al., methods are identical (*i.e.* based on the solution of the exact adjoint problem). Therefore it is a clean and essential comparison.

3 Sections

Section 1.

(a) It can be done in the revised version. For the reference, since it is a rather general statement, we could cite for instance "K. M. Cuffey and W. S. B. Paterson. The Physics of Glaciers. Academic Press, 2010.". and more precisely the introduction of chapter 7 on basal slip.

Section 2.

(a) The Paterson's *Physics of glaciers* can be given as an ice flow modelling reference, possibly with Hutter (1983).

(b) It does seem a little premature to give these finite elements specifications in the first sentence. However, since it is part of the model description, we could move it to the end of subsection 2.1 along with the fixed point algorithm. The hyphenation of finite element shall be removed in the revised version.

(c) It will be done in the revised version. The definition for the *nt* subscript is as follows:

We introduce $(\boldsymbol{t}, \boldsymbol{n})$, the tangent-normal pair of unit vectors such that:

$$\underline{\boldsymbol{\sigma}} = (\underline{\boldsymbol{\sigma}} \cdot \boldsymbol{n})\boldsymbol{n} + (\underline{\boldsymbol{\sigma}} \cdot \boldsymbol{t})\boldsymbol{t} \tag{1}$$

with:

$$\underline{\boldsymbol{\sigma}} \cdot \boldsymbol{n} = \sigma_{nn}\boldsymbol{n} + \sigma_{nt}\boldsymbol{t} \quad , \quad \underline{\boldsymbol{\sigma}} \cdot \boldsymbol{t} = \sigma_{tn}\boldsymbol{n} + \sigma_{tt}\boldsymbol{t} \tag{2}$$

(d) Since the reference is still not available and the calculation of this Poiseuille-like solution for a power law Stokes flow in a flat channel is fairly simple, it can be given in a short appendix or a reference to the first author PhD thesis can be given instead.

(e) The reliability of Newton method for power-law flows is arguable. Although, it is true that the Newton method provides a very good rate of convergence in close-to-linear cases (power-law exponent n close to 1), this rate of convergence strongly deteriorates with increasing n (for instance $n = 3$ leads to a superlinear convergence but rather far from the theoretical quadratic rate). In addition, the *local* aspect of the Newton method can be a strong issue and the radius of its convergence disk is unknown. As a matter of fact, numerical tests shows that a rather good first-guess (typically obtained from the fixed point method) is required to obtain a converged solution. The slowly convergence of the fixed point method remains an important issue that we actually addressed in the following adjacent work: *Four-field finite element solver for quasi-Newtonian flows and variational sensitivity analysis, N. Martin and J. Monnier, SIAM-Journal on Scientific Computing, in minor revision*. In this work, a new algorithm for the solution of the power-law Stokes problem is built. This algorithm provides better computational time with low memory needs. An archive version can be provided.
In the present context, it seemed unnecessary to go into those technical details since the aim is to address the precision and efficiency of the inverse problem using any iterative algorithm and automatic differentiation. For this purpose, the fixed point method seemed the simplest and most generic approach.

(f) An illustration of the geometry and notation can be added in the revised version

(g) There are several unfortunate choice of notation on our side leading to this unclear situation for n. The Glen's flow law exponent n is the one defined in equation (3) and it is the same one used in equation (6) since this Poiseuille-like solution is depending on the exponent of the power-law rheology. The n appearing in equations (19), (26) and (31) should be renamed, for example N, as it just represent the upper-value for the summation and is equal to 3 for the three higher frequencies which are added to the carrier wave.

(h) Since the only control variable used in this paper is the friction coefficient β, the additional notation of $\boldsymbol{k}$ can be avoided. But in a more general description of the adjoint model, the vector $\boldsymbol{k}$ can include many input parameters (such as the reference viscosity η_0 or the power-law exponent n for example). On the other hand, β is a vector since it is a function of x in every simulation (although it is taken constant for the gradient test runs).

(i) We tried to be careful about that but we were not able to find an official rule for the transcription of Russian names in latin alphabet. A quick look on internet provides three major transcriptions which are Tikhonov, Tykhonov and Tychonoff. However the one you are proposing appears to be the most frequent one and can replace the previous one.

(j) The $\frac{1}{2}$ is missing before the misfit term (see equation (25)). However, all the values for γ provided in the paper are based on the definition without the $\frac{1}{2}$ so it would be a bit complicated to modify.

(k) It is, indeed, an unnecessary reference to the PhD thesis and the reference to Vogel is sufficient

(l) Since β is an output of the minimization algorithm and γ is an input, it seemed correct to write $j(\beta) = j(\beta;\gamma)$ using the semicolon notation. Though, the notation $j(\beta;\gamma)$ can be used everywhere in the revised version to be more consistent.

(m) In fact, the precise comparison is made between the cost $j(\beta;\gamma)$ and the best theoretical reachable cost $j(\beta_t;0)$ where β_t represents the target β.

(n) The terminology is used here to designates the fact that we are using synthetic data. However, noise is added every time time in the presented "twin experiments". Similarly, in section 3.1, the convergence of the direct solver is limited to simulate a limited precision on the data (p.3865 lines 1-10). We could not find the origin of the terminology "twin experiment".

(o) This is a mistake and rounding errors should be removed from the list in the revised version.

(p) Since section 3 is dedicated to the assessment of the adjoint code precision using the gradient test and since the most common gradient test uses order one finite difference scheme (because it is less computationally expensive), it seemed important to give the formula we are actually using. And as you recall, given the audience, it might be helpful. Sections 2.5.1 and 2.5.2 could be presented in paragraph or combined instead of being numbered. Regarding the reference, it does not seem necessary to cite a book for an order two centered finite difference scheme (it is more than classical). For the precise gradient test procedure, although it is a quite standard method for validating an adjoint code, it is rather difficult to find a reference describing this procedure elsewhere than in software manuals or documentations so it seemed fare to cite the documentation of our own software. However, the book of Gunzburger does deal with these problematics and the reference could be added.

Section 3.

(a) It is probably a poor choice of words. The "current model" is simply supposed to designate "our model". It could be replaced by "The model considered here" or "The present model".

(b) The introduction of section 3 explains the whole procedure. It is not an artifact but the way automatic differentiation handles iterative routines. This technique is called reverse accumulation. The given references (for instance, Christianson (1994) or Griewank (1989)) explain quite well this procedure. It is not dependent of the linear or non-linear aspect of the forward problem but only on the iterative aspect of its solve.

(c) Regarding the actual terms being neglected in terms of continuous adjoint equations, some more attention can indeed be paid. It is undoubtly an interesting aspect that does not appear in the first verion of this work.

Omitting the lateral boundaries, the adjoint system writes (see *e.g.* Petra et al., "A Newton method for inversion in a nonlinear Stokes ice sheet model "):

$$-\boldsymbol{div}(\underline{\boldsymbol{\Sigma}}) = 0 \text{ in } \Omega \tag{3}$$

$$\boldsymbol{div}(\mathbf{v}) = 0 \text{ in } \Omega \tag{4}$$

$$\underline{\boldsymbol{\Sigma}}\boldsymbol{n} = \mathbf{u}_s^{obs} - \mathbf{u} \text{ on } \Gamma_s \tag{5}$$

$$\boldsymbol{\Sigma}_{nt} = \beta^{1/m}\left(|\mathbf{u}_\tau|^{\frac{1-m}{m}}\mathbf{v}_\tau + (m-1)|\mathbf{u}_\tau|^{\frac{1-3m}{m}}(\mathbf{u}_\tau \otimes \mathbf{u}_\tau)\mathbf{v}_\tau\right) \text{ on } \Gamma_{fr} \tag{6}$$

$$\mathbf{v}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma_{fr} \tag{7}$$

where $\mathbf{v}$ denotes the adjoint velocity and $\boldsymbol{\Sigma}$ the adjoint stress tensor is defined by:

$$\underline{\boldsymbol{\Sigma}} = 2\eta(\mathbf{u}, n)\left(\mathrm{I} + \frac{1-n}{n}\frac{\underline{\boldsymbol{D}}(\mathbf{u}) \otimes \underline{\boldsymbol{D}}(\mathbf{u})}{\|\underline{\boldsymbol{D}}(\mathbf{u})\|_F^2}\right)\underline{\boldsymbol{D}}(\mathbf{v}) - \boldsymbol{Id}\, q \tag{8}$$

with q denoting the adjoint pressure, I the fourth-order identity tensor applied to order two tensors, $\boldsymbol{Id}$ the second order identity tensor and $'\otimes'$ the tensor product.

This problem is a linear problem in $\mathbf{v}$ and depends on the forward velocity $\mathbf{u}$.

The self-adjoint method consists in neglecting the non-linearity that it to say the dependence of the viscosity and, in the present situation the friction condition, on the solution $\mathbf{u}$. Equivalently, it corresponds to set $n = m = 1$ in the adjoint system (3)-(7). It is straightforward from the previous system to see that the corresponding adjoint operator, under this approximation (*i.e.* with $m = n = 1$), is in fact the forward Stokes operator for a Newtonian fluid and a linear friction, hence the "self-adjoint" terminology. This terminology is incorrect since, the forward velocity field $\mathbf{u}$ is generally computed from the non-linear full-Stokes solve and leads then to a velocity-independent yet spatially variable viscosity field and consequently to a non-symmetric problem (which cannot then represent a self-adjoint operator).

First, the non-linearity of the forward problem appears in the definition of the adjoint stress given in equation (8). The norm of the term $\dfrac{\underline{\boldsymbol{D}}(\mathbf{u}) \otimes \underline{\boldsymbol{D}}(\mathbf{u})}{\|\underline{\boldsymbol{D}}(\mathbf{u})\|_F^2}$ is simply one (since $\|\underline{\boldsymbol{D}} \otimes \underline{\boldsymbol{D}}\| = \|\underline{\boldsymbol{D}}\|_F \times \|\underline{\boldsymbol{D}}\|_F$

given a consistent choice of the fourth-order tensor norm with the Frobenius matrix norm). And the norm of the identity tensor is known to be greater or equal to one (and typically equal to one for the *sup* norm). It follows that the terms that are being dropped are comparable to the one that are kept in the "self-adjoint" approximation for $\frac{1-n}{n}$ close to one ($2/3$ for $n = 3$). It logically follows that the greater the non-linearity (the n), the greater the non-linear contribution.

The other non-linearity comes from the non-linear friction law and appears in equation (6). A similar calculation leasd to the exact same conclusion and for $m > 1$, the norm of the two terms following the friction coefficient β are comparable.

As pointed out by Reviewer #2 Stephen Cornford, this fact is well observed in the gradient test results performed for the "self-adjoint" approximation which provides a term $|I_\alpha - 1|$ always around 1.

As you suggested, these observations could really be of interest and support the results of our work so they could be included in the revised version.

(d) The $\frac{1}{2}$ is in place this time but the s of u_s^{obs} is missing. It will be corrected in the revised version.

(e) The two terminologies designate the same problem. Since the "backward" terminology appears only two times throughout the paper, it could just be replaced by the "adjoint" terminology in order to avoid any confusion.

(f) The gradient test normally requires unnoisy data in view of validation. With noisy data, the constant rate for the decreasing of the error could not be perfectly retrieved since the reference value provided by the finite difference would not match the computed one. In this case it helps to demonstrate that the incomplete reverse accumulation behaves well as the constante rate decay of the error remains the same for the different thresholds of the forward problem. It also shows that even with perfect data and a very good forward solution ($\nu = 10^{-8}$) the self-adjoint precision is bouded around 1 (see Section 3. (b) of the present letter).

(g) The solution of the exact adjoint problem still requires one assembly of a linear system and one solve whereas the self-adjoint method considers the tranpose of the last matrix of the non-linear loop of the forward problem as the adjoint one. If a factorized version of this matrix has been stored then the solution of the transpose problem can be extremely fast so it is still more expensive to solve the exact adjoint problem than to use the self-adjoint approximation but the difference will be much smaller in that case. A sentence can be added to make that point clear.

(h) see Introduction and Section 3. (b) of the present letter

(i) The direct solver designates the iterative solver used for the solution of the power-law Stokes problem which is here the fixed point method.

(j) It is indeed a matter of debate although the following references could be cited as actual surface velocity data providing this type of precision:

- I. Joughin, B. E. Smith, I. M. Howat, T. Scambos, and T. Moon. Greenland flow variability from ice-sheet-wide velocity mapping. Journal of Glaciology. 2010
- M. King. Rigorous GPS data-processing strategies for glaciological appli- cations. Journal of Glaciology. 2004
- E. Rignot, J. Mouginot, and B. Scheuchl. Ice flow of the antarctic ice sheet. Science. 2011

Section 4-6.

(a) there is no obvious relation involving subscript i so a more compact form could be to put the three frequencies in the same equation line. Another option could be to define three n_i providing the three wavelengths (for instance $2dx$, $4dx$ and $10dx$ for friction coefficient (19)). These friction coefficients are plotted on figure 4 for the three values of n and the reference to this figure is given on page 3869 line 15.

(b) They are defined by equation (19) as the indices of the discrete sum so they are indeed natural integers. Maybe the integer interval notation $[\![\quad]\!]$ could be used instead.

(c) It is due to the fact that for smaller noise the limitation of the "self-adjoint" precision becomes more and more visible whereas the exact adjoint behaves well. The smaller the noise, the more you need an accurate adjoint to reach the best precision you can expect. It is actually an important aspect of the present work since the proposed "incomplete adjoint method" aims to propose a method adjustable in precision (and therefore in terms of computational cost) according to the accuracy of the data instead of using only the "self-adjoint" as is regardless of its limitations.

(d) This equation contains two mistakes hence the possible confusion. Parameters n and n_t should be β and β_t (the target β). The same mistakes appears in the caption of Figure 5. The parameter L is not defined and corresponds to the horizontal length of the computational domain.

(e) The sentence is not very clear. Here is a possible reformulation of the paragraph p.3870 lines 22-26:
" Figure 5 clearly demonstrates the inability, for the "self-adjoint" method, to provide a gradient accurate enough for sufficiently low noise. For noise levels $\delta = 0.1\%$ and $\delta = 0.01\%$, the "self-adjoint" gradient does not allow the optimal misfit to be reached. Therefore, in these situations, the "self-adjoint" approximation is theoretically not valid. However, as we will see, the "self-adjoint" method shows a certain ability to retrieve the target parameter"

(f) It was supposed to mean "good identification" but it is unnecessary since the sentence speaks about the *best situation* so the word "quality"could simply be removed.

(g) It should be "Section 2.1" instead of equation (2.1)

(h) The beginning of the sentence should be something like "The trend of over-fitting one can observe on the final cost value for sufficiently small values of γ (..."

(i) It will be modified in the revised version

(j) The reverse accumulation process is not an artifact of the implementation (see Introduction and Section 3. (b) of the present letter). It is generally true that using an automatically differentiated adjoint leads to higher computational costs. However, we believe that a major strength of the automatic differentiation is its high flexibility and capability to provide the adjoint of complicated, non-linear systems without requiring to achieve the analytical derivation of the adjoint equations. The needs for data assimilation in geophysics become more and more tangible and, at the same time, numerical models become more and more complex and refined and the time required to derive and discretize adjoint equations could be a limitation that scientist cannot really overcome. On the other hand, our research interests have tried to focus on this computationnally extensive aspect through the derivation of a fast and low-memory full-Stokes solver (see Section 2. (e) of the present letter) and the incomplete adjoint approach proposed here also tries to take into account this concern.

(k) The revised version of the paper will include a more compact yet efficient notation for these equations.

(l) It will be modified in the revised version.

(m) The main goal of this work is to assess the accuracy and reliability of the "self-adjoint" method because it is a widely used approach and, since the exact adjoint for the full-Stokes problem only became available recently in the glaciological community (for example in our work), it seemed of importance to use this tool to infer its numerical validity. So the conclusion tries to weight the pros and cons of both method based on the results provided before. The main advantage of the "self-adjoint" method is the time-saving it provides in general. Although the CPU-time saving will be smaller (yet strictly positive as it is explained in section 3. (g) of the present letter) for a code implementing the discretization of the continuous adjoint equation, it could be precised that the other aspect of time saving would be the fact that it is straightforward to implement the "self-adjoint" approximation (which requires to transpose a matrix) and that the implementation of the exact adjoint solution is a much more complicated matter (whatever the method considered).

4 References

(a) To recall the "acronym"DassFlow. It can be modified

(b) It will be modified in the revised version (if the reference is maintained).

5 Language

Many remarks have been made by the reviewers regarding the language. A thorough attention to this aspect will be paid in the revised version of the paper in order to provide a higher level of english.