

Reviewer: Ed Waddington
Authors: Roberts et al.
Manuscript: "Refined broad-scale sub-glacial morphology ..."
Journal: *The Cryosphere*

I like the approach that Roberts et al. are taking to interpolate sparse ice-thickness information in Aurora Subglacial Basin. Even though the amount of glacier physics incorporated into the interpolation scheme is not exhaustive, nevertheless it does a much better job than traditional spatial interpolation schemes based on no physics.

Building on earlier work by Warner and Budd (2000), the basic idea is that steady-state volumetric ice flux in any flowband can be estimated kinematically from upstream area and accumulation rate. When ice is frozen to bedrock, and the Shallow Ice Approximation (SIA) is applicable, ice flux can also be estimated dynamically from the local ice thickness and slope. By knowing the flux kinematically, and knowing the local ice slope from altimetry, it is then possible to estimate the ice thickness.

Using ice physics is good, and the simplest historical way to introduce ice physics has been to assume that the ice is perfectly plastic with a yield stress τ_0 . In that case, by assuming that the yield stress is reached everywhere at the base, i.e. $\tau_0 = \rho g D \nabla S$, the depth $D(x,y)$ can be estimated as

$$D = [\tau_0 / (\rho g)] (\nabla S)^{-1}. \quad (1)$$

Only the surface slope is needed, and τ_0 (in the coefficient $[\tau_0 / (\rho g)]$) is the adjustable parameter that plays the role of c_{eff} in the current manuscript.

Has anyone attempted to infer bedrock using (1)? If so, does a simple plastic scheme perform better than an inverse-cubic law, i.e. are the major gains obtained by introducing at least some ice physics, or by then refining the physics?

By assuming the SIA, this current manuscript (and Warner and Budd, 2000) offer a more-sophisticated physics-based interpolation scheme. I would also be curious to see how much improvement the SIA approach makes in comparison to a plastic scheme. I suspect that that this improvement in sophistication also makes a big difference.

My only concern about the paper is the non-intuitive nature of the units for c_{eff} and $t(x,y)$. I have a bit of trouble gaining much insight from an "effective flow parameter" coefficient c_{eff} that has units of $\text{m}^{3/5} \text{yr}^{1/5}$, and a "thickness factor" that has units of $\text{m}^{2/5} \text{yr}^{-1/5}$.

I think the work could be made more accessible to readers by non-dimensionalizing the flux equation and expressing the adjustable parameter c_{eff} in terms of a traditional enhancement factor in Glen's flow law. Here's how I would suggest.

Glen's law for SIA is often written as

$$\dot{\epsilon}_{xz} = EA(\theta)\sigma_{xz}^n \quad (2)$$

where $\dot{\epsilon}_{xz}$ is the shear strain rate, E is the flow enhancement factor, θ is temperature, and σ_{xz} is shear stress. After setting shear stress to be

$$\sigma(x,y,z) = -\rho g(S-z)\nabla S \quad (3)$$

where $S(x,y)$ is surface elevation, and integrating from the bed a couple of times, the dynamic ice flux is found to be

$$q(x,y) = \frac{2EA_{eff}(\theta)}{n+2} (\rho g)^n D(x,y)^{n+2} |\nabla S|^{n-1} \nabla S \quad (4)$$

where $D(x,y)$ is the ice thickness. Essentially, Roberts et al. have re-arranged this equation to express D as a function of flux q and slope ∇S . Their equation (1) is equivalent to

$$D(x,y) = \left(\frac{(n+2)}{2EA_{eff}(\theta)(\rho g)^n} \right)^{1/(n+2)} \left(\frac{q(x,y)}{|\nabla S|^n} \right)^{1/(n+2)} \quad (5)$$

The first factor on the right becomes their c_{eff} , and the second factor becomes $t(x,y)$. (This is all old stuff straight out of Paterson. I just want to express the origin of their Equation (1) a little bit more clearly.)

If they were to nondimensionalize Equation (4) or (5) by choosing a characteristic ice thickness D_c , a characteristic surface slope α_c , and a characteristic temperature θ_c , then a characteristic horizontal scale L_c arises naturally from $\alpha_c = D_c/L_c$, and a characteristic ice softness A_c arises through $A_c = A(\theta_c)$. A characteristic ice flux also arises from

$$q_c = \left(\frac{2A_c(\rho g)^n}{(n+2)} \right)^{1/(n+2)} (\alpha_c)^n D_c^{n+2} = \left(\frac{2A_c(\rho g)^n}{(n+2)} \right)^{1/(n+2)} \left(\frac{D_c}{L_c} \right)^n D_c^{n+2} \quad (6)$$

Then write each dimensional variable in terms of the characteristic value (which carries the dimensions) and a nondimensional variable (indicated with a tilde) that should be of order unity, carrying the relative fluctuations around the characteristic values, i.e.

$$\begin{aligned} x &= L_c \tilde{x} & y &= L_c \tilde{y} \\ D(x,y) &= D_c \times \tilde{D}(\tilde{x}, \tilde{y}) \\ \nabla S(x,y) &= \alpha_c \times \tilde{\nabla S}(\tilde{x}, \tilde{y}) \\ \theta_{eff} &= \theta_c \times \tilde{\theta}_{eff} \\ A_{eff}(\theta) &= A_c \times \tilde{A}_{eff}(\tilde{\theta}) \\ q(x,y) &= q_c \times \tilde{q}(\tilde{x}, \tilde{y}) \end{aligned} \quad (7)$$

After substituting (6) into (3) or (4), collecting all the characteristic numbers, and recognizing that by the way it is defined, q_c cancels out the other characteristic numbers, we are left with

$$\tilde{D}(\tilde{x}, \tilde{y}) = \left(\frac{1}{E \tilde{A}_{eff}(\theta)} \right)^{1/(n+2)} \left[\frac{\tilde{q}(\tilde{x}, \tilde{y})}{|\nabla \tilde{S}(\tilde{x}, \tilde{y})|^n} \right]^{1/(n+2)} \quad (8)$$

Equation (8) has several advantages.

- The adjustable coefficient is clearly related to well-known glaciological parameters, the nondimensional flow enhancement factor E , and the softness parameter A in Glen's law.
- The appearance of $\tilde{A}_{eff}(\tilde{\theta}_c)$ in the coefficient more directly indicates that deviations from linearity (or indeed from a coefficient of unity) can also be caused by spatial differences in deep ice temperature. However, if one decides to overlook temperature differences in the deep ice across the basin (for now), then $\tilde{A}_{eff}(\tilde{\theta}_c) = 1$ everywhere, and the flow law enhance factor E is the only player.
- The variables on both sides are now dimensionless numbers that should vary around unity. If the SIA applies, if $\theta = \theta_c$, and if $E=1$, then a correlation between \tilde{D} vs $\left[\tilde{q}(\tilde{x}, \tilde{y}) / |\nabla \tilde{S}(\tilde{x}, \tilde{y})|^n \right]^{1/(n+2)}$ will have a 1:1 slope with zero intercept.
- It should be more transparent to readers that when the coefficient $c_{eff} = (E \tilde{A}_{eff})^{-1/5}$ differs from unity, the fitting procedure is sweeping up all discrepancies from SIA (basal sliding, transient flow, softer ice, etc) into the enhancement factor E .

If the editors and authors are convinced that this would be a clearer presentation format, I think that all that would need to be changed would be the figure labels. A few paragraphs should also be added to explain the goals and benefits of the nondimensionalization.

Here are some editorial details.

Page 659 line 1

“ ... data ... **were** first mapped ...”

Page 659 last line

The inverse cubic relation is introduced here without explanation or citation. Perhaps cite Lythe et al. (2001) here.

Page 659 line 1

“ ... data ... **were** first mapped ...”

Page 660, first 2 paragraphs.

Both high values of t and low values of t that fail to follow the straight line occur near ice ridges and domes. The fact that both extremes occur in the same generic terrains may deserve some additional comment beyond just attributing the failure to low slope in the first case, and low flux in the second case. Flux and slope are both low near all ridges and domes. Perhaps, more fundamentally, as Equation (8) (above) suggests, the failure arises because the SIA does not apply there? Or is it simply because the method can become unstable in slow flow in the presence of data errors, in spite of the $1/5$ power?

4 lines above 2.1

- misplaced “only”. Text should read “... these criteria **exclude only** ...”

Equation (2)

What happens if $t_i = t_p$ in the denominator?

Below (2)

“... summations range over all streamlines involving p and using ...”

Presumably p is on only one flowline. How do other flowlines get involved? It would be helpful to explain if some lateral averaging scheme is being introduced here.

Below (3)

- misplaced “only”. Text should read “... method **varies only** ...”

Section 3. line 2

“... data around the test point **are** excluded ...”

Page 663 2nd last line –

“... data from the flight lines **were** mapped ...”

Page 664, line 5

Needs hyphen “...simple **inverse-distance** cube method ...”

Page 664, line 26 –

Misplaced “only, and mis-spelled “increasing”. Text should read “... with the biases **increasing only** slowly ...”

Page 664, line 27 –

Should have hyphen in “... an **inverse-distance** method ...”

Tables 2, 3, and 4 –

It would be helpful if the captions explained or defined r and r^2 respectively.

Figure 3a

The figure seems to call for 3 straight lines, perhaps with nearly flat lines for small and large t ?