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Reformulating the full-Stokes ice sheet model for a more efficient computational solution

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Abstract

The first-order or Blatter-Pattyn ice sheet model is an attractive alternative to the full Stokes model in many applications because of its reduced computational demands, in spite of an approximate stress tensor and a limitation to small basal boundary slopes. In contrast, the full unapproximated Stokes ice sheet model is more difficult to solve and computationally more expensive. This is due to the fact that while both models arise from a variational principle, the Blatter-Pattyn variational functional is positive-definite and involves just the horizontal velocity components, while the Stokes functional is indefinite and involves all three velocity components, as well as the pressure. These unfavorable properties arise because Stokes flow is treated as a constrained minimization problem where the pressure acts as a Lagrange multiplier that enforces incompressibility or zero velocity divergence. To alleviate these problems we reformulate the full-Stokes problem as an unconstrained, positive-definite minimization problem, quite analogous to the Blatter-Pattyn model but without the associated approximations, by introducing a velocity field that is already divergence-free and satisfies appropriate boundary conditions, thus dispensing with the need for a pressure. Such a velocity field is obtained by vertically integrating the continuity equation to obtain the vertical velocity as a function of the horizontal velocity components, as is done in the Blatter-Pattyn model. This leads to a reduced system for just the horizontal velocity components, again just as in the Blatter-Pattyn model. We thus obtain not only a reformulated action principle, which itself is sufficient for obtaining an efficient discrete model, but also a novel set of Euler-Lagrange partial differential equations and boundary conditions that specify the Stokes problem in terms of just the horizontal velocities. The derivations are performed not only for the common case of an ice sheet in contact with and sliding along the bed, which again is analogous to the Blatter-Pattyn model, but also for more general situations, such as for a floating ice shelf.

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1 Introduction

The most general and accurate model currently used for the simulation of ice sheet dynamics is based on non-Newtonian Stokes flow (e.g., Greve and Blatter, 2009). At present, however, a full-Stokes model presents formidable challenges for large-scale ice sheet modeling, although such models exist and are being used (e.g., Zwinger and Moore, 2009, implemented in the ELMER (http://www.csc.fi/english/pages/elmer) code package). As a consequence, there is considerable interest in various approximate models (e.g., the first order or Blatter-Pattyn approximation, and the shallow ice and shallow shelf approximations) that are more limited but computationally far cheaper than the full-Stokes model (e.g., Pattyn et al., 2008).

In Dukowicz et al. (2010) and Dukowicz et al. (2011) (henceforth referred to as DPL1 and DPL2, respectively) it was shown that non-Newtonian Stokes flow, including the boundary conditions, may be expressed as a constrained variational principle expressed in terms of an action $A_S[u_i, P, \Lambda]$ whose the arguments represent the functions with respect to which a stationary point is to be found. These arguments are composed of $u_i \in \{u, v, w\}$, the components of the three-dimensional velocity vector, and two Lagrange multipliers, the "pressures" P and Λ, that are used to enforce incompressibility, $\partial u_i / \partial x_i = 0$, and the no-penetration constraint at a fixed, rigid basal surface, $u_i n_i = 0$, respectively. Here, $x_i \in \{x, y, z\}$ is the position vector, and n_i is the outward-pointing unit vector at the ice sheet bounding surfaces. Note that Cartesian tensor notation is being used, and, where appropriate, the summation convention on repeated indices. In general, tensor indices are three-dimensional, i.e., $i, j, \dots \in \{x, y, z\}$, except when an index appears in parentheses, in which case it denotes an index in the horizontal plane only, e.g., $(i), (j), \dots \in \{x, y\}$, so that, for example, $u_{(i)} \in \{u, v\}$, $u_i u_i = u^2 + v^2 + w^2$ and $u_{(i)} u_{(i)} = u^2 + v^2$.

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Neglecting Λ for the moment, the discrete Euler-Lagrange equations resulting from the variational principle may be expressed in matrix form as follows

$$\begin{bmatrix} \mathbf{A} & \mathbf{G} \\ \mathbf{G}^\mathrm{T} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_i \\ \boldsymbol{P} \end{bmatrix} = \begin{bmatrix} \boldsymbol{b}_i \\ \boldsymbol{q} \end{bmatrix}, \tag{1}$$

where $\mathbf{A} = \mathbf{A}^T$ is a square, symmetric, positive-definite matrix representing the negative of the discrete nonlinear stress divergence operator in the momentum equations, $\mathbf{G}$ is the negative of the discrete gradient operator, and $\mathbf{G}^T$ is the discrete divergence operator. The right hand side contains contributions from gravitational forces and boundary conditions. The system matrix on the left hand side of Eq. (1) is symmetric but indefinite, meaning that the eigenvalues are real but have both positive and negative values. The solution $[\boldsymbol{u}_i, \boldsymbol{P}]^T$ is therefore at a saddle point of the action $A_\mathrm{S}[u_i, P]$. This is an example of the so-called "saddle point" problem that typically arises, as in this case, from a constrained optimization problem. There are two main difficulties in the solution of such problems. First, the discretization of the action that resulted in Eq. (1), and also the choice of basis functions for the pressure and velocity in particular, have to be done carefully so that the discrete problem has a "good" solution (this involves satisfying the so-called Brezzi-Babuska condition). Secondly, large-scale saddle point problems are typically solved using Krylov subspace methods (conjugate gradient-type algorithms). Unfortunately, in this case such methods tend to converge slowly, and may even fail, so it is necessary to find and apply a good preconditioner to achieve reasonable convergence. In fact, there is a voluminous literature on appropriate methods for the numerical solution of saddle point problems (see Benzi et al., 2005, and Benzi and Wathen, 2008, for example).

In glaciology, these difficulties have typically been avoided by approximating the Stokes model to obtain the so-called first-order model, otherwise called the Blatter-Pattyn model, first introduced by Blatter (1995) and refined by Pattyn (2003). The Blatter-Pattyn model is obtained by assuming that the ratio of the characteristic vertical and horizontal length scales in the ice sheet velocity field is small, i.e., a small aspect

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ratio approximation, allowing for the approximation/neglect of the mixed horizontal-vertical stress tensor components (which has been justified by a rigorous scale analysis; see Schoof and Hindmarsh, 2010). As a result, it becomes possible to vertically integrate the vertical momentum equation to obtain pressure as a function of the vertical velocity, $P = P(w)$, and the continuity equation to obtain the vertical velocity as a function of the horizontal velocity components, $w = w\left(u_{(i)}\right)$ (see Pattyn, 2003, or DPL1). This allows the elimination of both the pressure and vertical velocity from the approximated Stokes model to obtain a reduced system in terms of the horizontal velocity components only. The resulting system is derivable from a positive-definite action $A_{BP}\left[u_{(i)}\right]$ (Schoof, 2010; DPL1), which may be expressed in matrix form as follows

$$\tilde{\mathbf{A}}\boldsymbol{u}_{(i)} = \boldsymbol{b}_{(i)}, \tag{2}$$

where $\tilde{\mathbf{A}} = \tilde{\mathbf{A}}^T$ is a square, symmetric, positive-definite matrix of reduced rank compared to matrix $\mathbf{A}$ in Eq. (1). In contrast to Eq. (1), the system corresponding to Eq. (2) represents the minimization of a positive-definite action and is therefore amenable to solution by Krylov subspace methods (Knoll and Keyes, 2004) or even by direct numerical optimization methods (Nocedal and Wright, 2006). Therefore, and also because of its reduced rank, the Blatter-Pattyn system, Eq. (2), is much easier to solve than the full Stokes system, Eq. (1). However, the Blatter-Pattyn model is more limited in applications than the full-Stokes model (e.g., see the discussion and results in Pattyn et al., 2008). This is because of the small aspect ratio approximation employed in the Blatter-Pattyn model, and, in addition, because of a further approximation implicitly built in to the Blatter-Pattyn model, limiting it to small basal slopes, $\left|\partial z_{\mathrm{b}} / \partial x_{(i)}\right| \ll 1$ (see DPL2).

In the present paper we make the observation that there is no need for the Lagrange multipliers P and Λ if one already has a velocity field that satisfies both continuity and the basal no-penetration boundary condition. Such a velocity field is available, at least in principle, from vertically integrating the continuity equation to obtain $w = w\left(u_{(i)}\right)$, as in the Blatter-Pattyn model. Substituting this into the Stokes action, $A_{\mathrm{S}}[u_i, P, \Lambda]$, we obtain the "reformulated Stokes" action, as follows

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$$A_{RS}\left[u_{(i)}\right] = A_S\left[u_{(i)}, w\left(u_{(i)}\right), P = 0, \Lambda = 0\right], \tag{3}$$

which, together with $w = w\left(u_{(i)}\right)$ forms a complete specification of the Stokes problem. Note the following properties: (a) the action $A_{RS}\left[u_{(i)}\right]$ is exactly equivalent to the Stokes action, as indicated in Eq. (3), (b) since both Lagrange multipliers are set to zero, the reformulated action is positive-definite, just as in the Blatter-Pattyn model, and (c) this action leads to a matrix system of exactly the same form as Eq. (2). The resulting matrix system, therefore, has exactly the same beneficial properties as the Blatter-Pattyn system, Eq. (2), except that now there are no approximations as there are in the Blatter-Pattyn system.

It is interesting to note that Pattyn (2008) presents a reformulation of the full-Stokes model that superficially also resembles the Blatter-Pattyn model. However, this reformulation basically amounts to expressing the pressure P in terms of an alternative variable, the vertical stress component τ_{zz}, and it leads to an iteration scheme that is effectively equivalent to the solution of the system in the form of Eq. (1).

In the remainder of the paper we review the basic Stokes problem in Sect. 2, making the simplifying assumption that the ice sheet is in contact with and sliding along a rigid, fixed bed, as in DPL2 and elsewhere in the literature. In Sect. 3 we generalize the basal boundary condition to allow for a moving basal surface and the possibility of mass flux across the surface, as at the base of a floating ice shelf, thus generalizing the expression for the vertical velocity, $w = w\left(u_{(i)}\right)$. In Sect. 4 we obtain the reformulated Stokes action $A_{RS}\left[u_{(i)}\right]$, and finally, in Sect. 5 we obtain the corresponding reformulated Euler-Lagrange partial differential equations and boundary conditions.

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2 The basic Stokes model

We begin with the variational principle for the non-Newtonian ice sheet Stokes model whose action functional (see DPL2) is given by

$$A_S[u_i,P,\Lambda] = \int_V dV\left(G_n\left(\dot{\varepsilon}^2\right) - \rho g_i u_i - P\frac{\partial u_i}{\partial x_i}\right) + \int_{S^{(b)}} dS\left(\Lambda u_i n_i - \Sigma_j(u) n_j\right), \quad (4)$$

where g_i is the gravitational acceleration vector, (typically $g_i = (0,0,-g)$), ρ is the ice density, assumed constant, and $\dot{\varepsilon}^2 = \dot{\varepsilon}_{ij}\dot{\varepsilon}_{ij}$ is the second invariant of the full Stokes strain-rate tensor,

$$\dot{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

such that, expanded in Cartesian coordinates, we have

$$\dot{\varepsilon}^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 + \frac{1}{2}\left[\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2\right]. \quad (5)$$

We define

$$G_n\left(\dot{\varepsilon}^2\right) = \frac{2n}{n+1}\mu_n\left(\dot{\varepsilon}^2\right)\dot{\varepsilon}^2, \quad (6)$$

where

$$\mu_n\left(\dot{\varepsilon}^2\right) = \mu_0(\theta)\left(\dot{\varepsilon}^2\right)^{(1-n)/2n}, \quad (7)$$

is the Glen's law viscosity coefficient, typically used with exponent $n = 3$, and $\mu_0(\theta)$ is a temperature-dependent coefficient. As mentioned previously, the functional, Eq. (4), represents a constrained minimization principle, with the constraints enforced by two Lagrange multipliers, P and Λ. As in DPL1 and DPL2, we illustrate the effect of basal

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stress forces by $\Sigma_j(u) = -\beta u_i u_i n_j/2$, which represents a linear frictional sliding law with a constant coefficient β; $\beta \geq 0$. However, other frictional laws are easily accommodated as in Schoof (2010), for example. The two integrals are assumed to be over the entire ice sheet volume and the basal surface, respectively.

The variational principle states that the solution of this dynamical system in terms of the arguments, i.e., the velocity components u_i and pressures P and Λ, is to be found at the stationary point of the action, Eq. (4), obtained by setting the functional derivatives with respect to the arguments equal to zero, as follows

$$\frac{\delta A_S}{\delta u_i} = 0, \quad \frac{\delta A_S}{\delta P} = 0, \quad \frac{\delta A_S}{\delta \Lambda} = 0. \tag{8}$$

This yields the following Euler-Lagrange equations:

a. a three-dimensional momentum equation,

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho g_i = \frac{\partial \tau_{ij}}{\partial x_j} - \frac{\partial P}{\partial x_i} + \rho g_i = 0, \tag{9}$$

where $\sigma_{ij} = \tau_{ij} - P\delta_{ij}$ is the Cauchy stress tensor and $\tau_{ij} = 2\mu_n(\dot{\varepsilon}^2)\dot{\varepsilon}_{ij}$ is the deviatoric stress tensor,

b. the continuity equation for incompressible flow,

$$\frac{\partial u_i}{\partial x_i} = 0, \tag{10}$$

and the following boundary conditions,

c. a stress-free boundary conditions over the upper surface $S^{(s)}$, specified at any instant of time by $z = z_s(x, y, t)$:

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$$\sigma_{ij} n_j^{(s)} = \tau_{ij} n_j^{(s)} - P n_i^{(s)} = \tau_{ij} n_j^{(s)} = 0, \tag{11}$$

i.e., setting $P = 0$ at the upper surface, and

d. a frictional sliding boundary condition along a rigid basal surface $S^{(b)}$, specified by $z = z_b(x, y)$:

$$u_i^{(b)} n_i^{(b)} = 0, \tag{12}$$

$$\tau_{ij} n_j^{(b)} - \left(n_k^{(b)} \tau_{kj} n_j^{(b)} \right) n_i^{(b)} + \beta u_i^{(b)} = 0. \tag{13}$$

The unit normal vectors that appear here are defined as follows

$$n_j^{(s)} = \left(n_x^{(s)}, n_y^{(s)}, n_z^{(s)} \right)^T = \frac{\left(-\partial z_s / \partial x, -\partial z_s / \partial y, 1 \right)^T}{\sqrt{1 + \left(\partial z_s / \partial x \right)^2 + \left(\partial z_s / \partial y \right)^2}}, \tag{14}$$

$$n_j^{(b)} = \left(n_x^{(b)}, n_y^{(b)}, n_z^{(b)} \right)^T = \frac{\left(\partial z_b / \partial x, \partial z_b / \partial y, -1 \right)^T}{\sqrt{1 + \left(\partial z_b / \partial x \right)^2 + \left(\partial z_b / \partial y \right)^2}}. \tag{15}$$

For clarity, we shall employ superscripts (s) and (b), and subscripts s and b, to indicate an upper surface or basal value, respectively, particularly in those cases where confusion is possible. For concreteness, we have assumed a simplified ice sheet configuration illustrated in Fig. 1 that is subject to boundary conditions, Eqs. (11–12), namely, an upper surface entirely exposed to the atmosphere and a basal surface that is entirely in contact with and sliding along a rigid bed. Further, for the purpose of this paper we implicitly define the upper surface by the condition $n_z^{(s)} > 0$, and the basal surface by $n_z^{(b)} < 0$. We have chosen to use this commonly

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used configuration since there is a great variety of possible configurations and boundary conditions, and it is impossible to deal with them all. The Stokes model itself is of course entirely general. In the next Section we shall indicate how to generalize to a moving and possibly melting basal surface, as at the base of a floating ice shelf.

3 Generalizing the basal boundary condition

So far we have assumed a fixed and rigid basal surface specified by $z = z_\mathrm{b}(x,y)$. In such a case the no-penetration condition, Eq. (12), is given by

$$w^{(\mathrm{b})} = u_{(i)}^{(\mathrm{b})} \frac{\partial z_\mathrm{b}}{\partial x_{(i)}}. \tag{16}$$

More generally, for a moving material surface (i.e., a Lagrangian surface with no inflowing or outflowing flux due to a gain or loss of mass crossing the surface) and specified by $z = z_\mathrm{b}(x,y,t)$, we have

$$w^{(\mathrm{b})} = \frac{\partial z_\mathrm{b}}{\partial t} + u_{(i)} \frac{\partial z_\mathrm{b}}{\partial x_{(i)}}. \tag{17}$$

In addition, assuming an *outward* flux of mass leaving the ice sheet at the basal surface with a normal velocity of magnitude u_n, which may be due to melting, ablation, etc., we obtain

$$w^{(\mathrm{b})} = w_n^{(\mathrm{b})} + u_{(i)} \frac{\partial z_\mathrm{b}}{\partial x_{(i)}}, \tag{18}$$

where $w_n^{(\mathrm{b})} = \partial z_\mathrm{b}/\partial t - u_\mathrm{n}\sqrt{1 + (\partial z_\mathrm{b}/\partial x)^2 + (\partial z_\mathrm{b}/\partial y)^2}$ is the effective net basal vertical velocity due to both the motion of the interface and an outflowing mass flux.

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Integrating the continuity equation, Eq. (10), in the vertical direction with Eq. (18) as the boundary condition, and using Leibniz's theorem, the vertical velocity is given by

$$w = w_n^{(b)} - \frac{\partial}{\partial x_{(i)}} \int_{z_b}^{z} u_{(i)} dz'. \tag{19}$$

This corresponds to the relation $w = w\left(u_{(i)}\right)$ referred to earlier. In general, and in particular at the base of a floating ice shelf, we might expect that $w_n^{(b)} \neq 0$. For our present purpose, however, we assume that it is a given quantity. In general, therefore, the velocity $w_n^{(b)}$ is unknown and must be determined by the simultaneous solution of the ice sheet problem and the external environment.

We note that $w_n^{(b)}$ will effectively vanish along certain sections of the ice sheet basal surface (i.e., when the ice sheet is sliding in contact with a fixed and rigid bed) and have nonzero values elsewhere. It may therefore be considered as a general function of the horizontal position vector $x_{(i)}$ over the entire basal surface. Similarly, the friction coefficient β may be considered as a function of horizontal position over the entire basal surface, vanishing when the ice sheet is no longer in contact with the bed. This way, the surface integral in Eq. (4) may be extended over the entire basal surface without loss of generality. However, in general we expect that β is zero when $w_n^{(b)}$ is nonzero and vice versa, so that $\beta w_n^{(b)} = 0$. In the following, we shall assume this to be true, while leaving open the possibility of exceptions under unusual circumstances.

4 The reformulated action principle

As discussed in Sect. 1, the Lagrangian multipliers P and Λ are no longer needed if the vertical velocity given by Eq. (19) is used in the action functional, Eq. (4). This is because the three-dimensional velocity field, given by the horizontal velocity components and the vertical velocity from Eq. (19), already satisfies the continuity equation,

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Eq. (10), and the correct basal boundary condition, Eq. (18). Substituting this velocity field into Eq. (5), the action, Eq. (4), now becomes a function of horizontal velocity only,

$$A_{\mathrm{RS}}\left[u_{(i)}\right]=\int_V dV\left[G_n\left(\dot{\varepsilon}^2_{\mathrm{RS}}\right)+\rho g w\left(u_{(i)}\right)\right]-\int_{S^{(\mathrm{b})}} dS\,\Sigma_j'''\left(u^{(\mathrm{b})}\right)n_j^{(\mathrm{b})}, \tag{20}$$

where

$$\begin{aligned}\dot{\varepsilon}^2_{\mathrm{RS}}=&\left(\tfrac{\partial u}{\partial x}\right)^2+\left(\tfrac{\partial v}{\partial y}\right)^2+\left(\tfrac{\partial u}{\partial x}+\tfrac{\partial v}{\partial y}\right)^2+\tfrac{1}{2}\left(\tfrac{\partial u}{\partial y}+\tfrac{\partial v}{\partial x}\right)^2\\&+\tfrac{1}{2}\left[\left(\frac{\partial u}{\partial z}+\frac{\partial w\left(u_{(i)}\right)}{\partial x}\right)^2+\left(\frac{\partial v}{\partial z}+\frac{\partial w\left(u_{(i)}\right)}{\partial y}\right)^2\right],\end{aligned} \tag{21}$$

$$w\left(u_{(i)}\right)=w_n^{(\mathrm{b})}-\frac{\partial}{\partial x_{(i)}}\int_{z_{\mathrm{b}}}^{z}u_{(i)}\,dz', \tag{22}$$

$$\Sigma_j'''\left(u^{(\mathrm{b})}\right)n_j^{(\mathrm{b})}=-\frac{1}{2}\beta\left[u_{(i)}^{(\mathrm{b})}u_{(i)}^{(\mathrm{b})}+\left(w_n^{(\mathrm{b})}+u_{(i)}^{(\mathrm{b})}\frac{\partial z_{\mathrm{b}}}{\partial x_{(i)}}\right)^2\right], \tag{23}$$

and note that we have made use of Eqs. (18) and (19). The subscript RS stands for "Reformulated Stokes". Observe that $\dot{\varepsilon}^2_{\mathrm{RS}}=\dot{\varepsilon}^2$ for any velocity field that satisfies the full Stokes equations, the continuity equation, and boundary condition, Eq. (18). In general, the term involving $w_n^{(\mathrm{b})}$ vanishes in Eq. (23) because of our assumption that $\beta w_n^{(\mathrm{b})}=0$. As shown in Appendix A, the gravitational term in Eq. (20) may be expanded and simplified, as follows

$$\int_V dV\,w\left(u_{(i)}\right)=\int_V dV\left[u_{(i)}\frac{\partial z_{\mathrm{s}}}{\partial x_{(i)}}+u_{(i)}^{(\mathrm{b})}\frac{\partial z_{\mathrm{b}}}{\partial x_{(i)}}\right]-\int_{S^{(\mathrm{b})}}n_{(i)}^{(\mathrm{b})}dS\left(z_{\mathrm{s}}-z_{\mathrm{b}}\right)u_{(i)}^{(\mathrm{b})}+\int_V dV\,w_n^{(\mathrm{b})}. \tag{24}$$

Note that the last term on the right hand side is independent of $u_{(i)}$; as such, it does not participate in the variational principle and so may be omitted. Substituting into

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Eq. (20), the action takes the following alternative and equivalent form,

$$A'_{\mathrm{RS}}\left[u_{(i)}\right] = \int_V dV \left[G_n\left(\dot{\varepsilon}^2_{\mathrm{RS}}\right) + \rho g \left(u_{(i)} \frac{\partial z_{\mathrm{s}}}{\partial x_{(i)}} + u^{(\mathrm{b})}_{(i)} \frac{\partial z_{\mathrm{b}}}{\partial x_{(i)}} \right) \right]$$
$$- \int_{S^{(\mathrm{b})}} dS \left[\Sigma'''_j\left(u^{(\mathrm{b})}\right) n^{(\mathrm{b})}_j + \rho g (z_{\mathrm{s}} - z_{\mathrm{b}}) n^{(\mathrm{b})}_{(i)} u^{(b)}_{(i)} \right]. \tag{25}$$

This functional (excluding the gravitational terms which are responsible for the forcing only) is a positive-definite quantity, as alluded to previously, in contrast to the standard Stokes functional. Therefore, the variational principle is now a true minimization problem subject to gravitational forcing, just as in the Blatter-Pattyn approximate model. Also, this is a three-dimensional problem in only two variables, i.e., the two horizontal velocity components, again as in the Blatter-Pattyn model. Furthermore, all boundary conditions are automatically and correctly incorporated, including the basal no-penetration (or tangential flow) boundary condition, if applicable. Note that this functional is to be used jointly with Eq. (22) to obtain the complete three-dimensional velocity field.

The action, Eq. (25), (or Eq. 20, but this alternative is less appealing) is the preferred starting point for a numerical solution of the problem. This is because the discretization of the variational principle applied to Eq. (25) automatically yields a symmetric, positive-definite matrix problem of reduced rank, analogous to Eq. (2), which is optimal for an efficient numerical solution, as discussed earlier. Nevertheless, it is also of interest to obtain the associated partial differential equations, if only to compare them with the standard Stokes system, Eqs. (9–13). For this we need to derive the associated Euler-Lagrange equations, which we do next.

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5 New Euler-Lagrange equations for the reformulated Stokes system

Taking the variation of the action, Eq. (25), as in DPL1, and making use of Eqs. (22) and (23), we obtain

$$\delta A'_{\mathrm{RS}} = \int_V dV \delta G_n\left(\dot{\varepsilon}^2_{\mathrm{RS}}\right) + \int_V dV \rho g \left(\delta u_{(i)} \frac{\partial z_{\mathrm{s}}}{\partial x_{(i)}} + \delta u^{(\mathrm{b})}_{(i)} \frac{\partial z_{\mathrm{b}}}{\partial x_{(i)}}\right)$$
$$+ \int_{S^{(\mathrm{b})}} dS \delta u^{(\mathrm{b})}_{(i)} \left[\beta u^{(\mathrm{b})}_{(i)} + \beta\left(w^{(\mathrm{b})}_n + u^{(\mathrm{b})}_{(j)} \frac{\partial z_{\mathrm{b}}}{\partial x_{(j)}}\right) \frac{\partial z_{\mathrm{b}}}{\partial x_{(i)}} - \rho g (z_{\mathrm{s}} - z_{\mathrm{b}}) n^{(\mathrm{b})}_{(i)}\right]. \qquad (26)$$

Note that this is linear in the velocity perturbations $\delta u_{(i)}$, $\delta u^{(\mathrm{b})}_{(i)}$, and implicitly in $\delta u^{(\mathrm{s})}_{(i)}$ also. Recall that the variational principle, i.e., Eq. (8), implies that the variation of the action, Eq. (26), must vanish for arbitrary velocity perturbations. Therefore, Eq. (26) must now be manipulated into a form such that the velocity perturbations are linear multipliers in the integrands. Since the velocity perturbations are arbitrary, the coefficients multiplying each of the velocity perturbations must vanish, and this gives the required set of Euler-Lagrange equations and also the associated natural boundary conditions. The manipulations required to put Eq. (26) into this form are rather complicated. We do this in Appendix A, and obtain

$$\delta A'_{\mathrm{RS}} = -\int_V dV\, \delta u_{(i)} \left[\frac{\partial \tilde{\tau}_{(i)j}}{\partial x_j} + \frac{\partial}{\partial x_{(i)}}\left(\int_z^{z_{\mathrm{s}}} dz' \frac{\partial \tilde{\tau}_{(j)z}}{\partial x_{(j)}}\right) - \tilde{\tau}^{(\mathrm{s})}_{(j)z} n^{(\mathrm{s})}_{(j)} \sqrt{1 + \frac{\partial z_{\mathrm{s}}}{\partial x_{(i)}} \frac{\partial z_{\mathrm{s}}}{\partial x_{(i)}}} - \rho g \frac{\partial z_{\mathrm{s}}}{\partial x_{(i)}}\right]$$
$$+ \int_{S^{(\mathrm{s})}} dS\, \delta u^{(\mathrm{s})}_{(i)} \left(\tilde{\tau}_{(i)j} n^{(\mathrm{s})}_j - \tilde{\tau}_{(j)z} n^{(\mathrm{s})}_{(j)} \frac{\partial z_{\mathrm{s}}}{\partial x_{(i)}} - n^{(\mathrm{s})}_{(i)} \tilde{\tau}^{(\mathrm{s})}_{(j)z} n^{(\mathrm{s})}_{(j)} \sqrt{1 + \frac{\partial z_{\mathrm{s}}}{\partial x_{(i)}} \frac{\partial z_{\mathrm{s}}}{\partial x_{(i)}}}\right) \qquad (27)$$
$$+ \int_{S^{(\mathrm{b})}} dS\, \delta u^{(\mathrm{b})}_{(i)} \left[\begin{array}{l} \tilde{\tau}_{(i)j} n^{(\mathrm{b})}_j + \beta u^{(\mathrm{b})}_{(i)} + \left(\beta w^{(\mathrm{b})}_n + \beta u^{(\mathrm{b})}_{(j)} \frac{\partial z_{\mathrm{b}}}{\partial x_{(j)}} + \tilde{\tau}_{(j)z} n^{(\mathrm{b})}_{(j)}\right) \frac{\partial z_{\mathrm{b}}}{\partial x_{(i)}} \\ + n^{(\mathrm{b})}_{(i)} \tilde{\tau}^{(\mathrm{s})}_{(j)z} n^{(\mathrm{s})}_{(j)} \left(\sqrt{1 + \frac{\partial z_{\mathrm{b}}}{\partial x_{(i)}} \frac{\partial z_{\mathrm{b}}}{\partial x_{(i)}}} - \sqrt{1 + \frac{\partial z_{\mathrm{s}}}{\partial x_{(i)}} \frac{\partial z_{\mathrm{s}}}{\partial x_{(i)}}}\right) \end{array}\right],$$

where

$$\tilde{\tau}_{(i)j} = \mu_n(\dot{\varepsilon}^2_{\mathrm{RS}}) \begin{bmatrix} 2\left(2\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) & \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \left(\frac{\partial u}{\partial z} + \frac{\partial w\left(u_{(i)}\right)}{\partial x}\right) \\ \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & 2\left(\frac{\partial u}{\partial x} + 2\frac{\partial v}{\partial y}\right) & \left(\frac{\partial v}{\partial z} + \frac{\partial w\left(u_{(i)}\right)}{\partial y}\right) \end{bmatrix}, \tag{28}$$

and where Eqs. (21) and (22) define $\dot{\varepsilon}^2_{\mathrm{RS}}$ and $w\left(u_{(i)}\right)$, respectively. Thus, the Euler-Lagrange equations are given by

$$\frac{\partial \tilde{\tau}_{(i)j}}{\partial x_j} + \left[\frac{\partial}{\partial x_{(i)}}\left(\int_z^{z_s} dz' \frac{\partial \tilde{\tau}_{(j)z}}{\partial x_{(j)}}\right) - \tilde{\tau}^{(\mathrm{s})}_{(j)z} n^{(\mathrm{s})}_{(j)} \sqrt{1 + \frac{\partial z_\mathrm{s}}{\partial x_{(i)}}\frac{\partial z_\mathrm{s}}{\partial x_{(i)}}}\right] = \rho g \frac{\partial z_\mathrm{s}}{\partial x_{(i)}}. \tag{29}$$

The associated free-stress upper surface boundary condition is

$$\tilde{\tau}_{(i)j} n^{(\mathrm{s})}_j - \tilde{\tau}_{(j)z} n^{(\mathrm{s})}_{(j)} \frac{\partial z_\mathrm{s}}{\partial x_{(i)}} - \left[n^{(\mathrm{s})}_{(i)} \tilde{\tau}^{(\mathrm{s})}_{(j)z} n^{(\mathrm{s})}_{(j)} \sqrt{1 + \frac{\partial z_\mathrm{s}}{\partial x_{(i)}}\frac{\partial z_\mathrm{s}}{\partial x_{(i)}}}\right] = 0, \tag{30}$$

and the generalized basal boundary condition becomes

$$\begin{aligned} &\tilde{\tau}_{(i)j} n^{(\mathrm{b})}_j + \beta u^{(\mathrm{b})}_{(i)} + \left[\left(\beta w^{(\mathrm{b})}_n + \beta u^{(\mathrm{b})}_{(j)} \frac{\partial z_\mathrm{b}}{\partial x_{(j)}} + \tilde{\tau}_{(j)z} n^{(\mathrm{b})}_{(j)}\right) \frac{\partial z_\mathrm{b}}{\partial x_{(i)}}\right] \\ &\quad + \left[n^{(\mathrm{b})}_{(i)} \tilde{\tau}^{(\mathrm{s})}_{(j)z} n^{(\mathrm{s})}_{(j)} \left(\sqrt{1 + \frac{\partial z_\mathrm{b}}{\partial x_{(i)}}\frac{\partial z_\mathrm{b}}{\partial x_{(i)}}} - \sqrt{1 + \frac{\partial z_\mathrm{s}}{\partial x_{(i)}}\frac{\partial z_\mathrm{s}}{\partial x_{(i)}}}\right)\right] = 0. \end{aligned} \tag{31}$$

As noted earlier, we may set $\beta w^{(\mathrm{b})}_n = 0$ in Eq. (31) except possibly under unusual circumstances. These are the partial differential equations and boundary conditions that constitute the reformulated Stokes problem. The basal boundary conditions include sliding along a rigid bed as well as a generalized floating boundary condition that may, for example, include conditions at the base of an ice shelf. The above equations are very similar to the corresponding Blatter-Pattyn equations (see DPL1) except for extra

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terms, which we enclose in square brackets for emphasis. These extra terms, in effect, convert the Blatter-Pattyn model into the full-Stokes problem.

6 Conclusions

We have presented a reformulation of the full Stokes problem for ice sheets that converts it from the standard constrained minimization formulation in five variables (u,v,w,P,Λ) to an unconstrained minimization in only two variables (u,v). This not only reduces the size of the problem but makes the problem much more tractable numerically. From the original indefinite "saddle point" problem we obtain a positive-definite problem amenable to a number of efficient solution techniques. In this respect, the reformulated problem is similar to the first-order or Blatter-Pattyn approximation for ice sheets, but without the associated approximation errors. Note that this development provides a further example of the usefulness of the fundamental action principle for ice sheets presented in DPL1 and DPL2.

These properties of the reformulated Stokes system lead to a discrete problem expected to be much easier to solve. As a result, it might also be expected to be cheaper to solve. However, the new system matrix is more complicated and less sparse, as can be seen from the presence of integrals and (effectively) fourth-order horizontal velocity derivatives in Eq. (29). It should be noted, however, that the JFNK method of Knoll and Keyes (2004) will likely be the preferred solution method, in which case only the functional, Eq. (25), is required (i.e., the system matrix is never actually formed) and so only second-order horizontal velocity derivatives are needed. Therefore, it is not obvious at this point how the computational costs will compare. In any case, this question is beyond the scope of the present paper and can only be answered when the method is implemented in practice.

Appendix A

Preliminaries

We shall be making frequent use of the following two results:

a. interchanging the order of integration,

$$\int_a^b dx\, g(z',x)\int_a^x dy\, h(z'',y) = \int_a^b dy\, h(z'',y)\int_y^b dx\, g(z',x). \tag{A1}$$

We have introduced dummy variables z', z'' as a reminder that variables other than x, y may be present. A useful special case is given when $g\left(z',x\right) = 1$, as follows

$$\int_a^b dx \int_a^x dy\, h(z'',y) = \int_a^b dy\, (b-y)h(z'',y). \tag{A2}$$

b. Leibniz's Theorem,

$$\begin{aligned}\frac{\partial}{\partial x}\int_{a(x)}^{b(x)} dy\, h(z',x,y) = \int_{a(x)}^{b(x)} dy\, \frac{\partial h\left(z',x,y\right)}{\partial x} + h(z',x,b(x))\frac{\partial b(x)}{\partial x}\\ -h(z',x,a(x))\frac{\partial a(x)}{\partial x}.\end{aligned} \tag{A3}$$

A1 The gravity term in the initial form of the reformulated action, Eq. (20)

The gravity term in Eq. (20) contributes to the forcing terms in the Stokes equations. Leaving out the constant factor ρg and making use of Eq. (22), it may be usefully simplified as follows

$$\int_V dV\, w\left(u_{(i)}\right) = \int_V dV\, w_n^{(\mathrm{b})} - \int_V dV \frac{\partial}{\partial x_{(i)}}\int_{z_\mathrm{b}}^{z} u_{(i)}\, dz', \tag{A4}$$

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where, making use of Leibniz's theorem, the last term on the right hand side becomes

$$\int_V dV \frac{\partial}{\partial x_{(i)}} \int_{z_b}^{z} dz'\, u_{(i)} = \int_A dA \int_{z_b}^{z_s} dz \left(\frac{\partial}{\partial x_{(i)}} \int_{z_b}^{z} dz'\, u_{(i)} \right)$$

$$= \int_A dA \int_{z_b}^{z_s} dz \int_{z_b}^{z} dz' \frac{\partial u_{(i)}}{\partial x_{(i)}} - \int_A dA \int_{z_b}^{z_s} dz\, u_{(i)}^{(b)} \frac{\partial z_b}{\partial x_{(i)}}. \tag{A5}$$

Now, making use of Eq. (A2) and temporarily introducing $\tilde{u}_i = (u, v, 0)^T$, an extended version of $u_{(i)}$, we have

$$\int_V dV \frac{\partial}{\partial x_{(i)}} \int_{z_b}^{z} dz'\, u_{(i)} = \int_A dA \int_{z_b}^{z_s} dz \left[(z_s - z) \frac{\partial u_{(i)}}{\partial x_{(i)}} - u_{(i)}^{(b)} \frac{\partial z_b}{\partial x_{(i)}} \right]$$

$$= \int_V dV \left[(z_s - z) \frac{\partial \tilde{u}_i}{\partial x_i} - u_{(i)}^{(b)} \frac{\partial z_b}{\partial x_{(i)}} \right]. \tag{A6}$$

Using the chain rule and applying Gauss' theorem, we finally obtain

$$\int_V dV \frac{\partial}{\partial x_{(i)}} \int_{z_b}^{z} dz'\, u_{(i)} = \int_V dV \left[\frac{\partial (z_s - z)\tilde{u}_i}{\partial x_i} - u_{(i)} \frac{\partial z_s}{\partial x_{(i)}} - u_{(i)}^{(b)} \frac{\partial z_b}{\partial x_{(i)}} \right]$$

$$= -\int_V dV \left[u_{(i)} \frac{\partial z_s}{\partial x_{(i)}} + u_{(i)}^{(b)} \frac{\partial z_b}{\partial x_{(i)}} \right] + \int_{S^{(b)}} n_{(i)}^{(b)} dS (z_s - z_b) u_{(i)}^{(b)}. \tag{A7}$$

This, together with Eq. (A4) may now be used to obtain Eq. (24), and hence the simpler form of the action, Eq. (25).

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A2 Derivations leading to the Euler-Lagrange equations

There now remain two terms in Eq. (26) that need to be manipulated into the required form, namely,

$$I_1 = \int_V dV \delta G_n \left(\dot{\varepsilon}^2_{\mathrm{RS}}\right) \quad \text{and} \quad I_2 = \int_V dV \rho g \delta u^{(\mathrm{b})}_{(i)} \frac{\partial z_{\mathrm{b}}}{\partial x_{(i)}}.$$

The first term, I_1, is by far the most complicated and we shall deal with it first. To do this we shall temporarily assume that the vertical velocity is an independent variable, as in the standard Stokes model, and therefore retain a three-dimensional velocity in the form $u_i \in \{u_{(i)}, w\}$. However, from Eqs. (18) and (19), and noting that $\delta w^{(\mathrm{b})}_n = 0$, we have

$$\delta w = -\frac{\partial}{\partial x_{(i)}} \int_{z_{\mathrm{b}}}^{z} \delta u_{(i)} dz', \quad \delta w^{(\mathrm{b})} = \delta u_{(i)} \frac{\partial z_{\mathrm{b}}}{\partial x_{(i)}}. \tag{A8}$$

Following the procedures in DPL1, we obtain

$$I_1 = \int_V dV \delta G_n \left(\dot{\varepsilon}^2_{\mathrm{RS}}\right) = \int_V dV \, \tilde{\tau}_{ij} \frac{\partial \delta u_i}{\partial x_j} = I_{11} + I_{12} + I_{13}, \tag{A9}$$

where

$$I_{11} = -\int_V dV \delta u_i \frac{\partial \tilde{\tau}_{ij}}{\partial x_j}, \quad I_{12} = \int_{S^{(\mathrm{s})}} dS \, \delta u^{(\mathrm{s})}_i \tilde{\tau}_{ij} n^{(\mathrm{s})}_j, \quad I_{13} = \int_{S^{(\mathrm{b})}} dS \, \delta u^{(\mathrm{b})}_i \tilde{\tau}_{ij} n^{(\mathrm{b})}_j, \tag{A10}$$

and

$$\tilde{\tau}_{ij} = \mu_n(\dot{\varepsilon}^2_{\mathrm{RS}}) \begin{bmatrix} 2\left(2\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) & \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & 2\left(\frac{\partial u}{\partial x} + 2\frac{\partial v}{\partial y}\right) & \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) & \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) & 0 \end{bmatrix}. \tag{A11}$$

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The basal surface integral I_{13} is the simplest; where, making use of Eq. (A8), it may be rewritten as follows

$$I_{13} = \int_{S^{(b)}} dS\, \delta u_{(i)}^{(b)} \tilde{\tau}_{(i)j} n_j^{(b)} + \int_{S^{(b)}} dS\, \delta w^{(b)}\, \tilde{\tau}_{(j)z} n_{(j)}^{(b)}$$
$$= \int_{S^{(b)}} dS\, \delta u_{(i)}^{(b)} \left(\tilde{\tau}_{(i)j} n_j^{(b)} + \frac{\partial z_b}{\partial x_{(i)}} \tilde{\tau}_{(j)z} n_{(j)}^{(b)} \right). \tag{A12}$$

The upper surface integral I_{12} is more complicated. It may be expanded and rewritten as follows

$$I_{12} = \int_{S^{(s)}} dS\, \delta u_{(i)}^{(s)} \tilde{\tau}_{(i)j} n_j^{(s)} + I_{121}. \tag{A13}$$

Making use of Leibniz's theorem and Eq. (A8), the integral I_{121} becomes

$$I_{121} = \int_{S^{(s)}} dS\, \delta w^{(s)}\, \tilde{\tau}_{(j)z} n_{(j)}^{(s)}$$
$$= -\int_{S^{(s)}} dS\, \tilde{\tau}_{(j)z} n_{(j)}^{(s)} \left(\int_{z_b}^{z_s} dz' \frac{\partial \delta u_{(i)}}{\partial x_{(i)}} + \delta u_{(i)}^{(s)} \frac{\partial z_s}{\partial x_{(i)}} - \delta u_{(i)}^{(b)} \frac{\partial z_b}{\partial x_{(i)}} \right)$$
$$= -\int_{S^{(s)}} dS\, \delta u_{(i)}^{(s)}\, \tilde{\tau}_{(j)z} n_{(j)}^{(s)} \frac{\partial z_s}{\partial x_{(i)}} + I_{1211} + I_{1212}, \tag{A14}$$

where

$$I_{1211} = -\int_{S^{(s)}} dS\, \tilde{\tau}_{(j)z} n_{(j)}^{(s)} \int_{z_b}^{z_s} dz' \frac{\partial \delta u_{(i)}}{\partial x_{(i)}}, \quad I_{1212} = \int_{S^{(s)}} dS\, \delta u_{(i)}^{(b)}\, \tilde{\tau}_{(j)z} n_{(j)}^{(s)} \frac{\partial z_b}{\partial x_{(i)}}. \tag{A15}$$

Finally, the volume integral I_{11} becomes

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$$
\begin{aligned}
I_{11} &= -\int_V dV\,\delta u_{(i)}\frac{\partial\tilde{\tau}_{(i)j}}{\partial x_j} - \int_V dV\,\delta w\frac{\partial\tilde{\tau}_{(j)z}}{\partial x_{(j)}} \\
&= -\int_V dV\,\delta u_{(i)}\frac{\partial\tilde{\tau}_{(i)j}}{\partial x_j} + \int_V dV\left(\frac{\partial}{\partial x_{(i)}}\int_{z_\mathrm{b}}^{z}\delta u_{(i)}\,dz'\right)\frac{\partial\tilde{\tau}_{(j)z}}{\partial x_{(j)}} \\
&= -\int_V dV\left(\delta u_{(i)}\frac{\partial\tilde{\tau}_{(i)j}}{\partial x_j} + \delta u_{(i)}^{(\mathrm{b})}\frac{\partial z_\mathrm{b}}{\partial x_{(i)}}\frac{\partial\tilde{\tau}_{(j)z}}{\partial x_{(j)}}\right) + \int_V dV\left(\int_{z_\mathrm{b}}^{z}\frac{\partial\delta u_{(i)}}{\partial x_{(i)}}dz'\right)\frac{\partial\tilde{\tau}_{(j)z}}{\partial x_{(j)}} \\
&= -\int_V dV\,\delta u_{(i)}\frac{\partial\tilde{\tau}_{(i)j}}{\partial x_j} + I_{111} + I_{112},
\end{aligned}
\tag{A16}
$$

where again we have used Leibniz's theorem and Eq. (A8), and where

$$
I_{111} = -\int_V dV\,\delta u_{(i)}^{(\mathrm{b})}\frac{\partial z_\mathrm{b}}{\partial x_{(i)}}\frac{\partial\tilde{\tau}_{(j)z}}{\partial x_{(j)}}, \qquad I_{112} = \int_V dV\left(\int_{z_\mathrm{b}}^{z}\frac{\partial\delta u_{(i)}}{\partial x_{(i)}}dz'\right)\frac{\partial\tilde{\tau}_{(j)z}}{\partial x_{(j)}}.
\tag{A17}
$$

The last integral, I_{112}, may be put in the appropriate form by interchanging the order of integration, as follows

$$
\begin{aligned}
I_{112} &= \int_A dA\left[\int_{z_\mathrm{b}}^{z_\mathrm{s}} dz\frac{\partial\tilde{\tau}_{(j)z}}{\partial x_{(j)}}\left(\int_{z_\mathrm{b}}^{z} dz'\frac{\partial\delta u_{(i)}}{\partial x_{(i)}}\right)\right] = \int_A dA\left[\int_{z_\mathrm{b}}^{z_\mathrm{s}} dz\frac{\partial\delta u_{(i)}}{\partial x_{(i)}}\left(\int_{z}^{z_\mathrm{s}} dz'\frac{\partial\tilde{\tau}_{(j)z}}{\partial x_{(j)}}\right)\right] \\
&= \int_V dV\,\frac{\partial\delta u_{(i)}}{\partial x_{(i)}}\left(\int_{z}^{z_\mathrm{s}} dz'\frac{\partial\tilde{\tau}_{(j)z}}{\partial x_{(j)}}\right).
\end{aligned}
\tag{A18}
$$

Now, temporarily expressing the velocity perturbation as a three-dimensional vector, $\delta u_j = (\delta u, \delta v, 0)^T$, and applying Gauss' theorem, we obtain

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$$I_{112} = \int_V dV \, \frac{\partial \delta u_j}{\partial x_j} \left(\int_z^{z_s} dz' \frac{\partial \tilde{\tau}_{(i)z}}{\partial x_{(i)}} \right)$$
$$= \int_V dV \, \frac{\partial}{\partial x_j} \left(\delta u_j \int_z^{z_s} dz' \frac{\partial \tilde{\tau}_{(i)z}}{\partial x_{(i)}} \right) - \int_V dV \, \delta u_j \frac{\partial}{\partial x_j} \left(\int_z^{z_s} dz' \frac{\partial \tilde{\tau}_{(i)z}}{\partial x_{(i)}} \right)$$
$$= \int_{S^{(b)}} dS \, \delta u_{(j)}^{(b)} n_{(j)}^{(b)} \int_{z_b}^{z_s} dz' \frac{\partial \tilde{\tau}_{(i)z}}{\partial x_{(i)}} - \int_V dV \, \delta u_{(j)} \frac{\partial}{\partial x_{(j)}} \left(\int_z^{z_s} dz' \frac{\partial \tilde{\tau}_{(i)z}}{\partial x_{(i)}} \right). \tag{A19}$$

Note that integrals I_2 and I_{111} are basically of the same form. Combining them, and noting from Eq. (15) that $\partial z_b / \partial x_{(i)} = n_{(i)}^{(b)} \sqrt{1 + \partial z_b / \partial x_{(i)} \partial z_b / \partial x_{(i)}}$, we obtain

$$I_3 = I_2 + I_{111} = \int_V dV \, \delta u_{(i)}^{(b)} \left(\rho g - \frac{\partial \tilde{\tau}_{(j)z}}{\partial x_{(j)}} \right) \frac{\partial z_b}{\partial x_{(i)}}$$
$$= \int_V dV \, n_{(i)}^{(b)} \delta u_{(i)}^{(b)} \left(\rho g - \frac{\partial \tilde{\tau}_{(j)z}}{\partial x_{(j)}} \right) \sqrt{1 + \frac{\partial z_b}{\partial x_{(i)}} \frac{\partial z_b}{\partial x_{(i)}}}$$
$$= \int_{z_b}^{z_s} dz \int_A dA \, n_{(i)}^{(b)} \delta u_{(i)}^{(b)} \left(\rho g - \frac{\partial \tilde{\tau}_{(j)z}}{\partial x_{(j)}} \right) \sqrt{1 + \frac{\partial z_b}{\partial x_{(i)}} \frac{\partial z_b}{\partial x_{(i)}}}. \tag{A20}$$

Now, noting that $dS = dA \sqrt{1 + \partial z_b / \partial x_{(i)} \partial z_b / \partial x_{(i)}}$ on the basal surface, and carrying through the integration with respect to z, this takes the final form

$$I_3 = \int_{z_b}^{z_s} dz \int_{S^{(b)}} dS \, n_{(i)}^{(b)} \delta u_{(i)}^{(b)} \left(\rho g - \frac{\partial \tilde{\tau}_{(j)z}}{\partial x_{(j)}} \right)$$
$$= \int_{S^{(b)}} dS \, \delta u_{(i)}^{(b)} n_{(i)}^{(b)} \left(\rho g (z_s - z_b) - \int_{z_b}^{z_s} dz \frac{\partial \tilde{\tau}_{(j)z}}{\partial x_{(j)}} \right). \tag{A21}$$

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There are only two integrals left, I_{111} and I_{112}. Converting from $\partial z_{\mathrm{b}}/\partial x_{(i)}$ to $n_{(i)}^{(\mathrm{b})}$, as before, we obtain

$$I_{112}=\int_{S^{(\mathrm{s})}} dS\,\delta u_{(i)}^{(\mathrm{b})}\,\tilde{\tau}_{(j)z}n_{(j)}^{(\mathrm{s})}\frac{\partial z_{\mathrm{b}}}{\partial x_{(i)}}=\int_{S^{(\mathrm{b})}} dS\,n_{(i)}^{(\mathrm{b})}\delta u_{(i)}^{(\mathrm{b})}\,\tilde{\tau}_{(j)z}^{(\mathrm{s})}n_{(j)}^{(\mathrm{s})}\sqrt{1+\frac{\partial z_{\mathrm{b}}}{\partial x_{(i)}}\frac{\partial z_{\mathrm{b}}}{\partial x_{(i)}}},\tag{A22}$$

where, since the integrand is a function of horizontal position $x_{(i)}$only, it is permissible to switch the surface of integration from the upper to the basal surface. For the last integral, making use of the fact that $dS=dA\sqrt{1+\partial z_{\mathrm{s}}/\partial x_{(i)}\partial z_{\mathrm{s}}/\partial x_{(i)}}$ on the upper surface, we have

$$\begin{aligned}
I_{111}&=-\int_{S^{(\mathrm{s})}} dS\,\tilde{\tau}_{(j)z}^{(\mathrm{s})}n_{(j)}^{(\mathrm{s})}\int_{z_{\mathrm{b}}}^{z_{\mathrm{s}}} dz'\,\frac{\partial \delta u_{(i)}}{\partial x_{(i)}}\\
&=-\int_{A} dA\,\tilde{\tau}_{(j)z}^{(\mathrm{s})}n_{(j)}^{(\mathrm{s})}\sqrt{1+\frac{\partial z_{\mathrm{s}}}{\partial x_{(i)}}\frac{\partial z_{\mathrm{s}}}{\partial x_{(i)}}}\int_{z_{\mathrm{b}}}^{z_{\mathrm{s}}} dz'\,\frac{\partial \delta u_{(i)}}{\partial x_{(i)}}\\
&=-\int_{V} dV\,\tilde{\tau}_{(j)z}^{(\mathrm{s})}n_{(j)}^{(\mathrm{s})}\sqrt{1+\frac{\partial z_{\mathrm{s}}}{\partial x_{(i)}}\frac{\partial z_{\mathrm{s}}}{\partial x_{(i)}}}\frac{\partial \delta u_{(i)}}{\partial x_{(i)}}.
\end{aligned}\tag{A23}$$

Now, applying the chain rule and Gauss' theorem, we obtain the final form,

$$\begin{aligned}
I_{111}&=-\int_{V} dV\,\frac{\partial}{\partial x_{(i)}}\left(\delta u_{(i)}\tilde{\tau}_{(j)z}^{(\mathrm{s})}n_{(j)}^{(\mathrm{s})}\sqrt{1+\frac{\partial z_{\mathrm{s}}}{\partial x_{(i)}}\frac{\partial z_{\mathrm{s}}}{\partial x_{(i)}}}\right)\\
&\quad+\int_{V} dV\,\delta u_{(i)}\frac{\partial}{\partial x_{(i)}}\left(\tilde{\tau}_{(j)z}^{(\mathrm{s})}n_{(j)}^{(\mathrm{s})}\sqrt{1+\frac{\partial z_{\mathrm{s}}}{\partial x_{(i)}}\frac{\partial z_{\mathrm{s}}}{\partial x_{(i)}}}\right)\\
&=-\int_{S^{(\mathrm{s})}} dS\,n_{(i)}^{(\mathrm{s})}\delta u_{(i)}^{(\mathrm{s})}\tilde{\tau}_{(j)z}^{(\mathrm{s})}n_{(j)}^{(\mathrm{s})}\sqrt{1+\frac{\partial z_{\mathrm{s}}}{\partial x_{(i)}}\frac{\partial z_{\mathrm{s}}}{\partial x_{(i)}}}
\end{aligned}$$

$$-\int_{S^{(b)}} dS\, n^{(b)}_{(i)} \delta u^{(b)}_{(i)} \tilde{\tau}^{(s)}_{(j)z} n^{(s)}_{(j)} \sqrt{1+\frac{\partial z_s}{\partial x_{(i)}}\frac{\partial z_s}{\partial x_{(i)}}}$$

$$+\int_V dV\, \delta u_{(i)} \frac{\partial}{\partial x_{(i)}} \left(\tilde{\tau}^{(s)}_{(j)z} n^{(s)}_{(j)} \sqrt{1+\frac{\partial z_s}{\partial x_{(i)}}\frac{\partial z_s}{\partial x_{(i)}}} \right). \tag{A24}$$

Since everything is now in the required form, we may combine Eqs. (A9–A24) to obtain Eq. (27) in Sect. 5.

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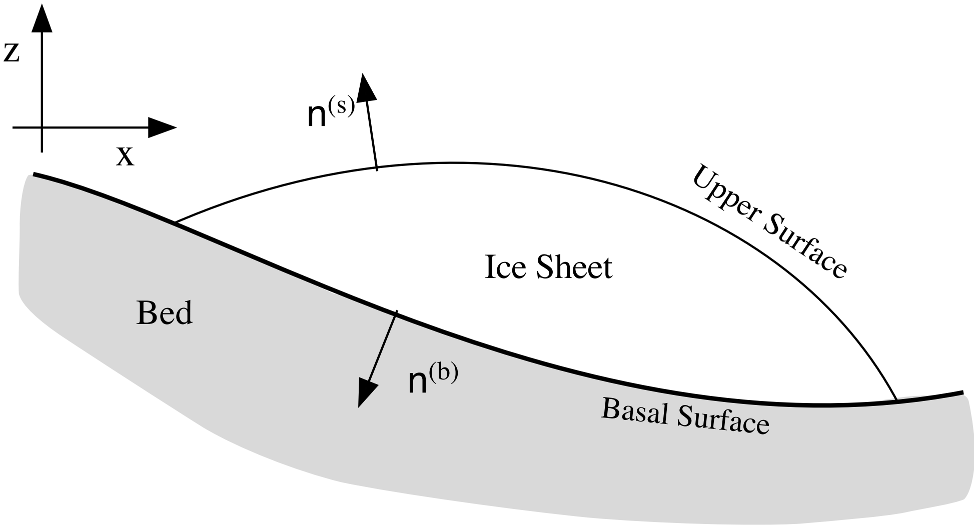

Fig. 1. A schematic diagram of the simplified ice sheet configuration discussed in Sect. 2.

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