The Cryosphere Discuss., 4, 495–560, 2010
www.the-cryosphere-discuss.net/4/495/2010/


The Cryosphere Discussions

This discussion paper is/has been under review for the journal The Cryosphere (TC). Please refer to the corresponding final paper in TC if available.



# Manufactured analytical solutions for isothermal full-Stokes ice sheet models

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Received: 25 March 2010 – Accepted: 27 March 2010 – Published: 8 April 2010

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Published by Copernicus Publications on behalf of the European Geosciences Union.

## Abstract

We present the detailed construction of an exact solution to time-dependent and steady-state isothermal full-Stokes ice sheet problems. The solutions are constructed for two-dimensional flowline and three-dimensional full-Stokes ice sheet models with variable viscosity. The construction is done by choosing for the specified ice surface and bed a velocity distribution that satisfies both mass conservation and the kinematic boundary conditions. Then a compensatory stress term in the conservation of momentum equations and their boundary conditions is calculated to make the chosen velocity distributions as well as the chosen pressure field into exact solutions. By substituting different ice surface and bed geometry formulas into the derived solution formulas, analytical solutions for different geometries can be constructed.

The boundary conditions can be specified as essential Dirichlet conditions or as periodic boundary conditions. By changing a parameter value, the analytical solutions allow investigation of algorithms for a different range of aspect ratios as well as for different, frozen or sliding, basal conditions. The analytical solutions can also be used to estimate the numerical error of the method in the case when the effects of the boundary conditions are eliminated, that is, when the exact solution values are specified as inflow and outflow boundary conditions.

## 1 Introduction

Model verification is crucial in developing a numerical model. The ice-sheet modeling community has been using two tools to verify models, comparison of numerically computed solutions to analytical solutions when possible, and *intercomparison*, that is, measuring differences between various models' results on the sets of simplified geometry benchmark tests.

For *shallow-ice approximation* (SIA) models, the simplified geometry tests as well as the results of intercomparison of different SIA models can be found in (Huybrechts et al., 1995). As for the exact solutions for SIA equations, two techniques have been used to generate analytical solutions, *the similarity reduction* technique (an approach that identifies equations for which the solution depends on certain groupings of the independent variables rather than depending on each of the independent variables separately (Nye, 2000; Halfar, 1981, 1983; Bueler et al., 2005) and *the manufactured solutions* technique (an approach that chooses a reasonable "solution" function, for example, a velocity-field and pressure, substitutes them into the Stokes equations, and determines the body force necessary to make the chosen functions into actual solutions (Bueler et al., 2005, 2007; Bueler and Brown, 2006).

For higher-order models and full-Stokes models, the simplified geometry tests and the results of intercomparison of different models can be found in (Pattyn et al., 2008). As for the exact solutions, mathematical work has mainly focused on the flow of *linear* media, and quasi-analytical solutions have been found for the first-order approximation equations for computing the three-dimensional stress and velocity field in grounded glaciers in (Blatter, 1995). Analytical solutions have been found describing transient two dimensional flow (Hutter, 1980, 1983; Johannesson, 1992), three-dimensional steady-state flow (Reeh, 1987; Johannesson, 1992) and transient evolution flow (Gudmundsson, 2003).

All the above solutions give physical insight into the flow processes; however, they cannot be easily used to benchmark the numerical solutions. For example, Gudmundsson in (Gudmundsson, 2003) obtained the three-dimensional solution of the linearized zeroth-order problem for a linear viscous medium. To use this solution for benchmarking numerical ice sheet models, the exact error estimate must be known (Raymond and Gudmundsson, 2005).

In this paper, we present the detailed construction of a manufactured exact solution to time-dependent and steady-state isothermal full-Stokes ice sheet problems. The solutions are constructed for three-dimensional (3-D) full-Stokes and two-dimensional (2-D) flowline ice sheet models with variable viscosity. The construction is done by choosing for the specified ice surface and bed the velocity distributions that satisfy the mass conservation equation and the kinematic boundary conditions, and by then calculating the required force distribution that makes the chosen velocities and pressure into exact solutions of the conservation of momentum equation and its boundary conditions. In the appendices we give the formulas that can be used to calculate the compensatory stress terms for the momentum equation in the 2-D and 3-D full-Stokes models and a fortran 77 code to calculate stress terms for the 2-D model.

The steady-state solutions constructed in this paper are variations of the benchmark experiments A and B in (Pattyn et al., 2008). However, by substituting different ice surface and bed geometry into the derived formulas, analytical solutions for different geometries can also be constructed.

The boundary conditions can be specified as essential Dirichlet conditions or as periodic boundary conditions. By changing a parameter value, the analytical solutions allow modelers to investigate their solutions for a range of aspect ratios as well as for different, frozen or sliding, basal conditions. Finally, the analytical solutions may help the modelers to estimate the numerical error in the case when the effect of the boundary conditions are eliminated, that is, when the exact solutions values are specified as inflow and outflow boundary conditions.

## 2 Model physics

### 2.1 Model equations

We consider an ice sheet model in the Cartesian coordinates $\tilde{\boldsymbol{x}} = (\tilde{x}, \tilde{y}, \tilde{z})$ with the domain $0 \le \tilde{x} \le L$, $0 \le \tilde{y} \le L$, $\tilde{b}(\tilde{x}, \tilde{y}) \le \tilde{z} \le \tilde{s}(\tilde{x}, \tilde{y}, \tilde{t})$, where $\tilde{t}$ is time, $\tilde{s}(\tilde{x}, \tilde{y}, \tilde{t})$ defines the surface and $\tilde{b}(\tilde{x}, \tilde{y})$ defines the base of the glacier.

Bed elevation $\tilde{b}(\tilde{x}, \tilde{y})$ and accumulation rate $\tilde{\dot{a}}$ are time independent, while surface elevation $\tilde{s}(\tilde{x}, \tilde{y}, \tilde{t})$ can change with time. The solution is the velocity vector $\tilde{\boldsymbol{v}} = (\tilde{u}, \tilde{v}, \tilde{w})$ and ice pressure $\tilde{p}$. Dimensional variables in this work are denoted with a tilde and non-dimensional variables without.

The field equations for the isothermal ice sheet model consist of the conservation of mass and the conservation of momentum:

$$\frac{\partial \tilde{u}}{\partial \tilde{x}}+\frac{\partial \tilde{v}}{\partial \tilde{y}}+\frac{\partial \tilde{w}}{\partial \tilde{z}}=0, \tag{1}$$

$$\frac{\partial\left(2\tilde{\mu}\frac{\partial \tilde{u}}{\partial \tilde{x}}+\tilde{p}\right)}{\partial \tilde{x}}+\frac{\partial\left(\tilde{\mu}\left(\frac{\partial \tilde{u}}{\partial \tilde{y}}+\frac{\partial \tilde{v}}{\partial \tilde{x}}\right)\right)}{\partial \tilde{y}}+\frac{\partial\left(\tilde{\mu}\left(\frac{\partial \tilde{u}}{\partial \tilde{z}}+\frac{\partial \tilde{w}}{\partial \tilde{x}}\right)\right)}{\partial \tilde{z}}=0, \tag{2}$$

$$\frac{\partial\left(\tilde{\mu}\left(\frac{\partial \tilde{u}}{\partial \tilde{y}}+\frac{\partial \tilde{v}}{\partial \tilde{x}}\right)\right)}{\partial \tilde{x}}+\frac{\partial\left(2\tilde{\mu}\frac{\partial \tilde{v}}{\partial \tilde{y}}+\tilde{p}\right)}{\partial \tilde{y}}+\frac{\partial\left(\tilde{\mu}\left(\frac{\partial \tilde{v}}{\partial \tilde{z}}+\frac{\partial \tilde{w}}{\partial \tilde{y}}\right)\right)}{\partial \tilde{z}}=0, \tag{3}$$

$$\frac{\partial\left(\tilde{\mu}\left(\frac{\partial \tilde{w}}{\partial \tilde{x}}+\frac{\partial \tilde{u}}{\partial \tilde{z}}\right)\right)}{\partial \tilde{x}}+\frac{\partial\left(\tilde{\mu}\left(\frac{\partial \tilde{w}}{\partial \tilde{y}}+\frac{\partial \tilde{v}}{\partial \tilde{z}}\right)\right)}{\partial \tilde{y}}+\frac{\partial\left(2\tilde{\mu}\frac{\partial \tilde{w}}{\partial \tilde{z}}+\tilde{p}\right)}{\partial \tilde{z}}=\tilde{\rho}\tilde{g}, \tag{4}$$

where $\tilde{\rho}$ is the ice density, $\tilde{g}$ is the gravitational acceleration, $\tilde{\mu}$ is the effective viscosity

$$\tilde{\mu}=\frac{B}{2}\left[\frac{1}{4}\left(\frac{\partial \tilde{u}}{\partial \tilde{y}}+\frac{\partial \tilde{v}}{\partial \tilde{x}}\right)^2+\frac{1}{4}\left(\frac{\partial \tilde{u}}{\partial \tilde{z}}+\frac{\partial \tilde{w}}{\partial \tilde{x}}\right)^2+\frac{1}{4}\left(\frac{\partial \tilde{v}}{\partial \tilde{z}}+\frac{\partial \tilde{w}}{\partial \tilde{y}}\right)^2 -\frac{\partial \tilde{u}}{\partial \tilde{x}}\frac{\partial \tilde{v}}{\partial \tilde{y}}-\frac{\partial \tilde{u}}{\partial \tilde{x}}\frac{\partial \tilde{w}}{\partial \tilde{z}}-\frac{\partial \tilde{v}}{\partial \tilde{y}}\frac{\partial \tilde{w}}{\partial \tilde{z}}\right]^{\frac{1-n}{2n}}, \tag{5}$$

$B$ is a temperature-independent rate factor, and $n$ is the stress exponent.

## 2.2 Boundary conditions

The model is time-dependent in the usual sense that the ice sheet geometry evolves according to a mass continuity equation. We assume that the ice has a hard bed, $\frac{\partial b}{\partial t}=0$. The kinematic boundary conditions applied at the upper and lower surfaces of the ice mass are as follows:

$$\frac{\partial \tilde{s}}{\partial \tilde{t}}+\tilde{u}(\tilde{x},\tilde{y},\tilde{s},\tilde{t})\frac{\partial \tilde{s}}{\partial \tilde{x}}+\tilde{v}(\tilde{x},\tilde{y},\tilde{s},\tilde{t})\frac{\partial \tilde{s}}{\partial \tilde{y}}-\tilde{w}(\tilde{x},\tilde{y},\tilde{s},\tilde{t})=\dot{\tilde{a}}, \tag{6}$$



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$$\tilde{u}(\tilde{x},\tilde{y},\tilde{b},\tilde{t})\frac{\partial \tilde{b}}{\partial \tilde{x}} + \tilde{v}(\tilde{x},\tilde{y},\tilde{b},\tilde{t})\frac{\partial \tilde{b}}{\partial \tilde{y}} - \tilde{w}(\tilde{x},\tilde{y},\tilde{b},\tilde{t}) = 0. \quad (7)$$

The stress-free boundary conditions at the upper surface $\tilde{s}(\tilde{x},\tilde{y},\tilde{t})$ are defined as:

$$\frac{1}{\tilde{r}_s}\left[-\frac{\partial \tilde{s}}{\partial \tilde{x}}\left(2\tilde{\mu}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \tilde{p}\right) - \frac{\partial \tilde{s}}{\partial \tilde{y}}\tilde{\mu}\left(\frac{\partial \tilde{u}}{\partial \tilde{y}} + \frac{\partial \tilde{v}}{\partial \tilde{x}}\right) + \tilde{\mu}\left(\frac{\partial \tilde{u}}{\partial \tilde{z}} + \frac{\partial \tilde{w}}{\partial \tilde{x}}\right)\right] = 0, \quad (8)$$

$$\frac{1}{\tilde{r}_s}\left[-\frac{\partial \tilde{s}}{\partial \tilde{x}}\tilde{\mu}\left(\frac{\partial \tilde{u}}{\partial \tilde{y}} + \frac{\partial \tilde{v}}{\partial \tilde{x}}\right) - \frac{\partial \tilde{s}}{\partial \tilde{y}}\left(2\tilde{\mu}\frac{\partial \tilde{v}}{\partial \tilde{y}} + \tilde{p}\right) + \tilde{\mu}\left(\frac{\partial \tilde{v}}{\partial \tilde{z}} + \frac{\partial \tilde{w}}{\partial \tilde{y}}\right)\right] = 0, \quad (9)$$

$$\frac{1}{\tilde{r}_s}\left[-\frac{\partial \tilde{s}}{\partial \tilde{x}}\left(\tilde{\mu}\left(\frac{\partial \tilde{w}}{\partial \tilde{x}} + \frac{\partial \tilde{u}}{\partial \tilde{z}}\right)\right) - \frac{\partial \tilde{s}}{\partial \tilde{y}}\left(\tilde{\mu}\left(\frac{\partial \tilde{w}}{\partial \tilde{y}} + \frac{\partial \tilde{v}}{\partial \tilde{z}}\right)\right) + \left(2\tilde{\mu}\frac{\partial \tilde{w}}{\partial \tilde{z}} + \tilde{p}\right)\right] = 0, \quad (10)$$

where $\tilde{r}_s = \sqrt{1 + \left(\frac{\partial \tilde{s}}{\partial \tilde{x}}\right)^2 + \left(\frac{\partial \tilde{s}}{\partial \tilde{y}}\right)^2}$.

For the frozen-based grounded ice, the boundary conditions at the bed $\tilde{b}(\tilde{x},\tilde{y})$ can be specified as Dirichlet conditions:

$$\tilde{u}(\tilde{x},\tilde{y},\tilde{b},\tilde{t}) = 0,$$
$$\tilde{v}(\tilde{x},\tilde{y},\tilde{b},\tilde{t}) = 0,$$
$$\tilde{w}(\tilde{x},\tilde{y},\tilde{b},\tilde{t}) = 0,$$
$$\tilde{p}(\tilde{x},\tilde{y},\tilde{b},\tilde{t}) = \tilde{\rho}\tilde{g}(\tilde{s} - \tilde{b}).$$

For the ice with sliding bed, the shear stresses may be specified at the bed $\tilde{b}(\tilde{x},\tilde{y})$ as Neumann conditions:

$$\frac{1}{\tilde{r}_b}\left[\frac{\partial \tilde{b}}{\partial \tilde{x}}\left(2\tilde{\mu}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \tilde{p}\right) + \frac{\partial \tilde{b}}{\partial \tilde{y}}\tilde{\mu}\left(\frac{\partial \tilde{u}}{\partial \tilde{y}} + \frac{\partial \tilde{v}}{\partial \tilde{x}}\right) - \tilde{\mu}\left(\frac{\partial \tilde{u}}{\partial \tilde{z}} + \frac{\partial \tilde{w}}{\partial \tilde{x}}\right)\right] = -\tilde{\beta}^2\tilde{u}(\tilde{x},\tilde{y},\tilde{b},\tilde{t}), \quad (11)$$

$$\frac{1}{\tilde{r}_b}\left[\frac{\partial \tilde{b}}{\partial \tilde{x}}\tilde{\mu}\left(\frac{\partial \tilde{u}}{\partial \tilde{y}} + \frac{\partial \tilde{v}}{\partial \tilde{x}}\right) + \frac{\partial \tilde{b}}{\partial \tilde{y}}\left(2\tilde{\mu}\frac{\partial \tilde{v}}{\partial \tilde{y}} + \tilde{p}\right) - \tilde{\mu}\left(\frac{\partial \tilde{v}}{\partial \tilde{z}} + \frac{\partial \tilde{w}}{\partial \tilde{y}}\right)\right] = -\tilde{\beta}^2\tilde{v}(\tilde{x},\tilde{y},\tilde{b},\tilde{t}), \quad (12)$$

$$\frac{1}{\tilde{r}_b}\left[\frac{\partial \tilde{b}}{\partial \tilde{x}}\left(\tilde{\mu}\left(\frac{\partial \tilde{w}}{\partial \tilde{x}} + \frac{\partial \tilde{u}}{\partial \tilde{z}}\right)\right) + \frac{\partial \tilde{b}}{\partial \tilde{y}}\left(\tilde{\mu}\left(\frac{\partial \tilde{w}}{\partial \tilde{y}} + \frac{\partial \tilde{v}}{\partial \tilde{z}}\right)\right) - \left(2\tilde{\mu}\frac{\partial \tilde{w}}{\partial \tilde{z}} + \tilde{p}\right)\right] = \tilde{\rho}\tilde{g}\tilde{h}, \quad (13)$$



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where $\tilde{r}_b = \sqrt{1 + \left(\frac{\partial \tilde{b}}{\partial \tilde{x}}\right)^2 + \left(\frac{\partial \tilde{b}}{\partial \tilde{y}}\right)^2}$ and $\tilde{\beta}^2$ is the friction coefficient.

Along the glacier's upstream and downstream boundaries, either periodic

$$\tilde{f}(0,\tilde{y},\tilde{z}) = \tilde{f}(L,\tilde{y},\tilde{z}),\ \frac{\partial \tilde{f}}{\partial \tilde{x}}(0,\tilde{y},\tilde{z}) = \frac{\partial \tilde{f}}{\partial \tilde{x}}(L,\tilde{y},\tilde{z});$$
$$\tilde{f}(\tilde{x},0,\tilde{z}) = \tilde{f}(\tilde{x},L,\tilde{z}),\ \frac{\partial \tilde{f}}{\partial \tilde{x}}(\tilde{x},0,\tilde{z}) = \frac{\partial \tilde{f}}{\partial \tilde{x}}(\tilde{x},L,\tilde{z});$$
$$where \tilde{f} = \tilde{u},\tilde{v},\tilde{w},\tilde{p},$$

or Dirichlet boundary conditions

$$\tilde{f}(i,\tilde{y},\tilde{z}) = \tilde{f}_{exact}(i,\tilde{y},\tilde{z}),\ i=0,L;$$
$$\tilde{f}(\tilde{x},j,\tilde{z}) = \tilde{f}_{exact}(\tilde{x},j,\tilde{z}),\ j=0,L;$$
$$where \tilde{f} = \tilde{u},\tilde{v},\tilde{w},\tilde{p}.$$

may be specified. Here we assume that functions $\tilde{f}_{exact}$ are known.

### 2.3 Dimensionless equations

To non-dimensionalize variables, we choose the following typical values: $Z$ – the mean thickness of the ice-sheet, $L$ – the length of ice-sheet, $U$ – a typical velocity in the horizontal direction, $W$ – a typical velocity in the vertical direction, $P$ – the mean pressure, $A$ – the mean accumulation/ablation rate, and introduce the following non-dimensional variables (variables without tilde):

$$\tilde{z} = Zz,\ \tilde{s} = Zs,\ \tilde{b} = Zb,$$
$$\tilde{x} = Lx,\ \tilde{y} = Ly,$$
$$\tilde{u} = Uu,\ \tilde{v} = Uv, \qquad (14)$$
$$\tilde{w} = Ww,$$



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$$\tilde{p} = Pp,$$
$$\tilde{t} = Tt,$$
$$\dot{\tilde{a}} = A\dot{a},$$
$$\tilde{\mu} = \frac{B}{2}\left(\frac{U}{L}\right)^{\frac{1-n}{n}}\mu.$$

To further simplify the equations, we introduce the aspect ratio parameter $\delta$:

$$\delta = \frac{Z}{L} \tag{15}$$

and require that scale factors $L$, $U$, $W$, and $P$ satisfy the following relationships:

$$\frac{B}{2}\left(\frac{U}{L}\right)^{\frac{1}{n}} = \tilde{\rho}\tilde{g}Z = P,\ \frac{WL}{UZ} = 1,\ T = \frac{Z}{W},\ A = W,\ \beta^2 = \frac{\tilde{\beta}^2 U}{P}. \tag{16}$$

The nondimensional steady-state conservation of mass and momentum equations are then as follows:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{17}$$

$$\delta\frac{\partial\left(2\mu\frac{\partial u}{\partial x} + p\right)}{\partial x} + \delta\frac{\partial\left(\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right)}{\partial y} + \frac{\partial\left(\mu\left(\frac{1}{\delta}\frac{\partial u}{\partial z} + \delta\frac{\partial w}{\partial x}\right)\right)}{\partial z} = 0, \tag{18}$$

$$\delta\frac{\partial\left(\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right)}{\partial y} + \delta\frac{\partial\left(2\mu\frac{\partial v}{\partial y} + p\right)}{\partial x} + \frac{\partial\left(\mu\left(\frac{1}{\delta}\frac{\partial v}{\partial z} + \delta\frac{\partial w}{\partial y}\right)\right)}{\partial z} = 0, \tag{19}$$

$$\delta\frac{\partial\left(\mu\left(\delta\frac{\partial w}{\partial x} + \frac{1}{\delta}\frac{\partial u}{\partial z}\right)\right)}{\partial x} + \delta\frac{\partial\left(\mu\left(\delta\frac{\partial w}{\partial y} + \frac{1}{\delta}\frac{\partial v}{\partial z}\right)\right)}{\partial y} + \frac{\partial\left(2\mu\frac{\partial w}{\partial z} + p\right)}{\partial z} - 1 = 0, \tag{20}$$

where

$$\mu = \left[\frac{1}{4}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \frac{1}{4}\left(\frac{1}{\delta}\frac{\partial u}{\partial z} + \delta\frac{\partial w}{\partial x}\right)^2 + \frac{1}{4}\left(\frac{1}{\delta}\frac{\partial v}{\partial z} + \delta\frac{\partial w}{\partial y}\right)^2 \right. \dots$$



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$$-\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial x}\frac{\partial w}{\partial z}-\frac{\partial v}{\partial y}\frac{\partial w}{\partial z}\Bigg]^{\frac{1-n}{2n}}. \tag{21}$$

The kinematic boundary conditions are invariant under the chosen set of scalings:

$$\frac{\partial s}{\partial t}+u(x,y,s(x,y,t),t)\frac{\partial s}{\partial x}+v(x,y,s(x,y,t),t)\frac{\partial s}{\partial y}-w(x,y,s(x,y,t),t)=\dot{a}, \tag{22}$$

$$u(x,y,b(x,y),t)\frac{\partial b}{\partial x}+v(x,y,b(x,y),t)\frac{\partial b}{\partial y}-w(x,y,b(x,y),t)=0. \tag{23}$$

The stress-free boundary conditions at the upper surface $s(x,y,t)$ become as follows:

$$\frac{1}{r_s}\left[-\delta\frac{\partial s}{\partial x}\left(2\mu\frac{\partial u}{\partial x}+p\right)-\delta\frac{\partial s}{\partial y}\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\mu\left(\frac{1}{\delta}\frac{\partial u}{\partial z}+\delta\frac{\partial w}{\partial x}\right)\right]=0, \tag{24}$$

$$\frac{1}{r_s}\left[-\delta\frac{\partial s}{\partial x}\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)-\delta\frac{\partial s}{\partial y}\left(2\mu\frac{\partial v}{\partial y}+p\right)+\mu\left(\frac{1}{\delta}\frac{\partial v}{\partial z}+\delta\frac{\partial w}{\partial y}\right)\right]=0, \tag{25}$$

$$\frac{1}{r_s}\left[-\delta\frac{\partial s}{\partial x}\left(\mu\left(\delta\frac{\partial w}{\partial x}+\frac{1}{\delta}\frac{\partial u}{\partial z}\right)\right)-\delta\frac{\partial s}{\partial y}\left(\mu\left(\delta\frac{\partial w}{\partial y}+\frac{1}{\delta}\frac{\partial v}{\partial z}\right)\right)+\left(2\mu\frac{\partial w}{\partial z}+p\right)\right]=0, \tag{26}$$

where $r_s=\sqrt{1+\delta^2\left(\frac{\partial s}{\partial x}\right)^2+\delta^2\left(\frac{\partial s}{\partial y}\right)^2}$.

The Neumann boundary conditions at the lower surface $b(x,y)$ become as follows:

$$\frac{1}{r_b}\left[\delta\frac{\partial b}{\partial x}\left(2\mu\frac{\partial u}{\partial x}+p\right)+\delta\frac{\partial b}{\partial y}\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)-\mu\left(\frac{1}{\delta}\frac{\partial u}{\partial z}+\delta\frac{\partial w}{\partial x}\right)\right]=-\beta^2 u, \tag{27}$$

$$\frac{1}{r_b}\left[\delta\frac{\partial b}{\partial x}\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\delta\frac{\partial b}{\partial y}\left(2\mu\frac{\partial v}{\partial y}+p\right)-\mu\left(\frac{1}{\delta}\frac{\partial v}{\partial z}+\delta\frac{\partial w}{\partial y}\right)\right]=-\beta^2 v, \tag{28}$$

$$\frac{1}{r_b}\left[\delta\frac{\partial b}{\partial x}\left(\mu\left(\delta\frac{\partial w}{\partial x}+\frac{1}{\delta}\frac{\partial u}{\partial z}\right)\right)+\delta\frac{\partial b}{\partial y}\left(\mu\left(\delta\frac{\partial w}{\partial y}+\frac{1}{\delta}\frac{\partial v}{\partial z}\right)\right)-\left(2\mu\frac{\partial w}{\partial z}+p\right)\right]=1, \tag{29}$$

where $r_b=\sqrt{1+\delta^2\left(\frac{\partial b}{\partial x}\right)^2+\delta^2\left(\frac{\partial b}{\partial y}\right)^2}$.

In scaled units, the glacier thickness and length are equal to unity.



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## 3 Manufactured analytical solutions of the 2-D full-Stokes isothermal flowline ice sheet model

### 3.1 Deriving an exact solution

Two-dimensional full-Stokes flowline models have only one horizontal dimension, $x$. So all terms in the Eqs. (18–29) that have variables $y$ or $v$, as well as all partial $y-$ derivatives of velocities and pressure can be removed.

To satisfy the 2-D version of the kinematic boundary conditions (22–23), we assume that in the interior of the domain, where $s(x,t) > b(x)$, the vertical velocity $w$ is

$$w(x,z,t) = u(x,z,t)\left(\frac{db}{dx}\frac{s-z}{s-b} + \frac{\partial s}{\partial x}\frac{z-b}{s-b}\right) + \left(\frac{\partial s}{\partial t} - \dot{a}\right)\frac{z-b}{s-b}. \tag{30}$$

From Eq. (30), it follows that

$$\frac{\partial w}{\partial z} = \frac{\partial u}{\partial z}\left(\frac{db}{dx}\frac{s-z}{s-b} + \frac{\partial s}{\partial x}\frac{z-b}{s-b}\right) + u\frac{\frac{\partial s}{\partial x} - \frac{db}{dx}}{s-b} + \frac{\frac{\partial s}{\partial t} - \dot{a}}{s-b}. \tag{31}$$

If we substitute (31) into the incompressibility Eq. (17), we obtain the following equation containing only variable $u$ and its derivatives:

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\left(\frac{db}{dx}\frac{s-z}{s-b} + \frac{\partial s}{\partial x}\frac{z-b}{s-b}\right) + u\frac{\frac{\partial s}{\partial x} - \frac{db}{dx}}{s-b} + \frac{\frac{\partial s}{\partial t} - \dot{a}}{s-b} = 0. \tag{32}$$

Equation (32) is a first-order quasi-linear partial differential equation with two independent variables ($x$ and $z$) and one dependent variable ($u$). The system of ordinary differential equations

$$\frac{dx}{1} = \frac{dz}{\frac{db}{dx}\frac{s-z}{s-b} + \frac{\partial s}{\partial x}\frac{z-b}{s-b}} = -\frac{du}{u\frac{\frac{\partial s}{\partial x} - \frac{db}{dx}}{s-b} + \frac{\frac{\partial s}{\partial t} - \dot{a}}{s-b}} \tag{33}$$



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is called the *characteristic system* of Eq. (32). If we can find two particular independent solutions of this system, which are called the *integrals of system* (33), in the form

$$\phi(x,z,u) = c_1, \ \psi(x,z,u) = c_2, \tag{34}$$

where $c_1$ and $c_2$ are arbitrary constants, then the general solution of Eq. (32) can be written as

$$\theta(\phi,\psi) = 0, \tag{35}$$

where $\theta$ is an arbitrary function of two variables. With Eq. (35) solved for $\phi$, the general solution can be written in the form

$$\phi = \vartheta(\psi), \tag{36}$$

where $\vartheta$ is an arbitrary function of one variable.

Thus, to solve Eq. (32), we have to find integrals $\phi$ and $\psi$ of the system (33). The first integral of the system (33) can be found by solving equation

$$\frac{dx}{1} = -\frac{du}{u\frac{\frac{\partial s}{\partial x}-\frac{db}{dx}}{s-b}+\frac{\frac{\partial s}{\partial t}-\dot{a}}{s-b}}. \tag{37}$$

Equation (37) can be re-written as follows:

$$\frac{du}{dx}+\frac{\frac{\partial s}{\partial x}-\frac{db}{dx}}{s-b}u = -\frac{\frac{\partial s}{\partial t}-\dot{a}}{s-b}. \tag{38}$$

We multiply both sides of Eq. (38) by $s-b$ and recognize that the left side of the equation is now the following product rule, $(s-b)\frac{\partial u}{\partial x}+\left(\frac{\partial s}{\partial x}-\frac{db}{dx}\right)u = \frac{\partial[u(s-b)]}{\partial x}$. After replacing the left side of the equation with this product rule, we obtain:

$$\frac{\partial[u(s-b)]}{\partial x} = -\frac{\partial s}{\partial t}+\dot{a}. \tag{39}$$



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Equation (39) has a solution

$$u \cdot (s-b) = -\int \left(\frac{\partial s}{\partial t} - \dot{a}\right) dx + c_1, \text{ or } c_1 = u \cdot (s-b) + \int \left(\frac{\partial s}{\partial t} - \dot{a}\right) dx, \tag{40}$$

where $c_1$ is a constant.

The second integral of the system (33) can be found by solving equation

$$\frac{dx}{1} = \frac{dz}{\frac{db}{dx}\frac{s-z}{s-b} + \frac{\partial s}{\partial x}\frac{z-b}{s-b}} \tag{41}$$

Equation (41) can be re-written as:

$$\frac{dz}{dx} - \frac{\frac{\partial s}{\partial x} - \frac{db}{dx}}{s-b} z = -\frac{\frac{\partial s}{\partial x} b - \frac{db}{dx} s}{s-b}, \tag{42}$$

After multiplying both sides of Eq. (42) by $\frac{1}{s-b}$, the equation can be transformed into:

$$\frac{d}{dx}\left(\frac{z}{s-b}\right) = \frac{d}{dx}\left(\frac{b}{s-b}\right). \tag{43}$$

Equation (43) has a solution

$$\frac{z}{s-b} = \frac{b}{s-b} + c_2, \text{ or } c_2 = \frac{z-b}{s-b}, \tag{44}$$

Thus, the general solution of Eq. (32) can be written as

$$\theta\left(u \cdot (s(x,t) - b(x)) + \int \left(\frac{\partial s}{\partial t} - \dot{a}\right) dx, \frac{z - b(x)}{s(x,t) - b(x)}\right) = 0, \tag{45}$$

where $\theta$ is an arbitrary function of two variables. With Eq. (45) solved for $u$, the general solution can be written in the form

$$u(x,z,t) = \frac{1}{s(x,t) - b(x)} \vartheta\left(\frac{z - b(x)}{s(x,t) - b(x)}\right) - \frac{1}{s(x,t) - b(x)} \int \left(\frac{\partial s}{\partial t} - \dot{a}\right) dx, \tag{46}$$

where $\vartheta$ is an arbitrary function of one variable.

The formula (46) shows that the functions satisfying the kinematic boundary conditions (22–23) and the conservation of mass Eq. (17), derived under assumption (30), depend on the form of the function $\vartheta$ and ice surface and bed curves.

Choose function $\vartheta$ as follows:

$$\vartheta(x) = c_x \left[1-(1-x)^\lambda\right] + c_b, \tag{47}$$

where $\lambda$, $c_x$, and $c_b$ are constants. The first term on the right-hand side of (47) may be considered as component of velocity associated with internal deformation, and $c_b$ as the basal sliding velocity coefficient.

Then the velocity field satisfying the 2-D versions of the kinematic boundary conditions (22–23) and the conservation of mass Eq. (17) is:

$$u(x,z,t) = \frac{c_x}{s-b}\left[1-\left(\frac{s-z}{s-b}\right)^\lambda\right] + \frac{c_b}{s-b} - \frac{1}{s-b}\int\left(\frac{\partial s}{\partial t} - \dot{a}\right)dx, \tag{48}$$

$$w(x,z,t) = u(x,z,t)\left(\frac{db}{dx}\frac{s-z}{s-b} + \frac{\partial s}{\partial x}\frac{z-b}{s-b}\right) + \left(\frac{\partial s}{\partial t} - \dot{a}\right)\frac{z-b}{s-b}, \tag{49}$$

For a zero-accumulation ($\dot{a}=0$) steady-state ($\frac{\partial s}{\partial t}=0$) flow with frozen bed ($c_b=0$), the horizontal velocity scaled to the surface velocity can be written as a function of ice scaled depth $d=\frac{s-z}{s-b}$:

$$u(x,z,t) = u(x,s,t)\left[1-\left(\frac{s-z}{s-b}\right)^\lambda\right] = u(x,s,t)\left[1-d^\lambda\right]. \tag{50}$$

This expression shows that the horizontal velocity from internal deformation increases with power $\lambda$ of ice depth. For $\lambda=4$ this is consistent with lamellar flow (der Veen, 1999) as shown in Fig. 1.

In addition to velocities, the ice pressure function should also be constructed. The manufactured solution for the ice pressure can be chosen, for example, as in Pattyn's higher-order model (Pattyn, 2003):

$$\tilde{p} = \sigma'_{\tilde{x}\tilde{x}} - \tilde{\rho}\tilde{g}(\tilde{s} - \tilde{z}) = 2\tilde{\mu}\frac{\partial \tilde{u}}{\partial \tilde{x}} - \tilde{\rho}\tilde{g}(\tilde{s} - \tilde{z}),$$

or in nondimensional form:

$$p(x,z,t) = 2\mu\frac{\partial u}{\partial x} - (s - z). \tag{51}$$

The constructed velocity and pressure functions do not necessarily satisfy the conservation of momentum Eqs. (18–20) or the surface and basal boundary conditions (24–26) and (27–29). To make the constructed velocity and pressure functions into exact solutions of these equations, we substitute them into the equations and calculate the right-hand side functions that match these solutions. This can be done when a specific surface $s(x,t)$ and bed $b(x)$ are chosen.

Equations (48–49) and (51) are solutions of flow with a general surface $s(x,t)$ and bed $b(x)$. Below are specific solutions for a particular case of an ice surface and a sinusoidal bed, similar to the benchmark experiment B in (Pattyn et al., 2008).

### 3.2 A manufactured solution for a time-dependent flow with a sinusoidal bed

To generate a particular solution, assume a flow with zero accumulation/ablation rate, $\dot{a} = 0$, a sinusoidal bed defined as in (Pattyn et al., 2008), and an ice surface that changes from a linear sloping surface to the one that is draped over the topography of the bed:

$$s(x,t) = s_0(x) + \eta(x)\gamma(t),\ s_0(x) = -x \cdot \tan(\alpha), \tag{52}$$

$$b(x) = s_0(x) - 1 + \eta(x), \tag{53}$$

where

$$\eta(x) = \frac{1}{2}\sin(2\pi x),\ \gamma(t) = 1 - e^{-c_t t},\ c_t \text{ is a constant}. \tag{54}$$

For a flow down an infinite plane with a mean inclination $\tan(\alpha)$, periodic boundary conditions for a function $f$ are defined as follows: $f(0, z + \tan(\alpha)) = f(1, z)$ and the analytical solutions (48), (49), (51) satisfy these conditions for geometry (52–53).

Appendices A and C1 contain the formulas and a simple fortran 77 code that can be used to calculate the exact solutions and compensatory stress terms for the momentum equation in the 2-D flowline model. The code dumps the generated solutions to specified files. All input data are specified in file `parameter 2d.h`.

Parameters of the flow are chosen as follows: the starting linear slope of the ice surface $\alpha = 0.5°$, coefficient in (54) $c_t = 10^{-6}$, and the constants in (48) $c_x = 10^{-6}$, $c_b = 10^{-6}$, and $\lambda = 4$. This experiment can be considered as an ice-stream flow over a bumpy bed. The values of constants $c_x$, $c_b$, and $c_t$ chosen to generate a reasonable dimensional values of the flow functions which calculated from nondimensional values using formulas (14). Values of flow parameters and constants are chosen from (Pattyn et al., 2008) and are given in Table (1). The length scale of the domain is chosen 80 km, which results in aspect ratio $\delta = \frac{1}{80}$. Velocity is shown in km/a and pressure in kPa.

Figure 2 shows the bed (53) and transformation over time of the ice surface of equation (52) (left graph) and the transformation of the norm of the surface velocity over time (right graph). The ice surface changes from a linear sloping surface to the surface draped over the topography of the bed. Ice thickness is spatially uniform when the steady-state solution is reached. The surface velocity at the beginning is anti-correlated with the ice thickness – it is larger over the bump than over the trough. At the steady-state, the surface velocity is spatially uniform and does not depend on the bed topography.

Figures 3, 4, and 5 show the horizontal velocity, vertical velocity, and pressure at the beginning and at the time when the steady-state solution is reached. The ice pressure is proportional to the ice thickness. At the steady-state, it is equal zero at the ice surface.

### 3.3 A steady-state manufactured solution for a flow with a linear sloping surface and a sinusoidal bed

To generate a steady-state solution, assume that in (52) the function $\gamma(t) = 0$, that is, a linear sloping surface and a sinusoidal bed are defined similar to the ones of the benchmark experiment B in (Pattyn et al., 2008):

$$s(x) = -x \cdot \tan(\alpha), \tag{55}$$

$$b(x) = s(x) - 1 + \frac{1}{2}\sin(2\pi x). \tag{56}$$

If we substitute the above functions for bed and surface into (48–49), then the corresponding steady-state flow's velocities are as follows:

$$u(x,z) = \frac{c_x}{1-\frac{1}{2}\sin(2\pi x)}\left[1-\left(\frac{-z-x\tan(\alpha)}{1-\frac{1}{2}\sin(2\pi x)}\right)^{\lambda}\right] + \frac{c_b}{1-\frac{1}{2}\sin(2\pi x)}, \tag{57}$$

$$w(x,z) = u(x,z)\left(\frac{db}{dx}\frac{s-z}{s-b} + \frac{ds}{dx}\frac{z-b}{s-b}\right). \tag{58}$$

Choice of coefficient $c_b = 0$ generates frozen bed flow with zero basal velocities, while $c_b \neq 0$ generates flow with a sliding bed.

As can be seen from (57–58), if $\lambda > 0$ then

at $z = b$, $u(x,b) = 0$, $w(x,b) = 0$;

at $z = s$, $u(x,s) = \dfrac{c_x}{s-b} = \dfrac{c_x}{h}$, $w(x,s) = \dfrac{\frac{ds}{dx}}{h}$.

The last expression shows the conservation of mass flux, $q = hu = c_x = \text{constant}$. This anti-correlated relationship between horizontal velocity and ice thickness is consistent with the simulation of the smallest length scale $L = 5$ km Experiment B in (Pattyn et al., 2008), by *all* full-Stokes models.

Figures 7 and 8 show the horizontal and vertical velocity, ice pressure, and the norm of the surface velocity corresponding to the flow with a linear sloping surface with a slope $\alpha = 0.5^{\circ}$ and a *frozen* sinusoidal bed ($c_b = 0$). The constants in (57) are chosen as $c_x = 10^{-6}$ and $\lambda = 4.0$ and the aspect ratio $\delta = \frac{1}{80}$.

Figure 9 shows the compensatory stresses $\Sigma_x$ and $\Sigma_z$ in the conservation of momentum equations calculated for the aspect ratio $\delta = \frac{1}{80}$. The graphs show that both stresses have largest values above the bump.

## 4 Analytical manufactured solutions of the 3-D isothermal full-Stokes ice-flow model

Assume as in the 2-D case that in the interior of the domain, $s(x,y,t) > b(x,y)$, the vertical velocity $w$ is:

$$w(x,y,z,t) = u(x,y,z,t)\left(\frac{\partial b}{\partial x}\frac{s-z}{s-b} + \frac{\partial s}{\partial x}\frac{z-b}{s-b}\right) + v(x,y,z)\left(\frac{\partial b}{\partial y}\frac{s-z}{s-b} + \frac{\partial s}{\partial y}\frac{z-b}{s-b}\right) + \left(\frac{\partial s}{\partial t} - \dot{a}\right)\frac{z-b}{s-b}, \tag{59}$$

then the kinematic boundary conditions (22–23) are satisfied. From (59), it follows that

$$\frac{\partial w}{\partial z} = \frac{\partial u}{\partial z}\left(\frac{\partial b}{\partial x}\frac{s-z}{s-b} + \frac{\partial s}{\partial x}\frac{z-b}{s-b}\right) + u\frac{\frac{\partial s}{\partial x} - \frac{\partial b}{\partial x}}{s-b} + \frac{\partial v}{\partial z}\left(\frac{\partial b}{\partial y}\frac{s-z}{s-b} + \frac{\partial s}{\partial y}\frac{z-b}{s-b}\right) + v\frac{\frac{\partial s}{\partial y} - \frac{\partial b}{\partial y}}{s-b} + \frac{1}{s-b}\left(\frac{\partial s}{\partial t} - \dot{a}\right). \tag{60}$$

If we substitute (60) into the incompressibility Eq. (17), we obtain the following equation containing only variables $u$, $v$ and their derivatives:

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\left(\frac{\partial b}{\partial x}\frac{s-z}{s-b} + \frac{\partial s}{\partial x}\frac{z-b}{s-b}\right) + u\frac{\frac{\partial s}{\partial x} - \frac{\partial b}{\partial x}}{s-b} \tag{61}$$

$$+\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\left(\frac{\partial b}{\partial y}\frac{s-z}{s-b}+\frac{\partial s}{\partial y}\frac{z-b}{s-b}\right)+v\frac{\frac{\partial s}{\partial y}-\frac{\partial b}{\partial y}}{s-b}$$
$$+\frac{1}{s-b}\left(\frac{\partial s}{\partial t}-\dot{a}\right)=0.$$

Equation (61) is a first-order quasi-linear partial differential equation with three independent variables ($x$, $y$, and $z$) and two dependent variables ($u$ and $v$) of type:

$$F\left(x,y,z,u(x,y,z,t),v(x,y,z,t),\frac{\partial u}{\partial x},\frac{\partial u}{\partial z},\frac{\partial v}{\partial y},\frac{\partial v}{\partial z}\right)=0. \tag{62}$$

Similar to the 2-D flowline manufactured solutions, we choose velocity $u(x,y,z,t)$ as the following function:

$$u(x,y,z,t)=c_x(s-b)^{\gamma_1}\left[1-\left(\frac{s-z}{s-b}\right)^{\lambda_1}\right]+c_{bx}\frac{1}{s-b}, \tag{63}$$

or

$$u(x,y,z,t)=c_x h^{\gamma_1}\left(1-d^{\lambda_1}\right)+c_{bx}\frac{1}{h}, \tag{64}$$

where $\gamma_1$, $\lambda_1$, $c_x$, $c_{bx}$ are constants, $d(x,y,z,t)=\frac{s-z}{s-b}$ is scaled ice depth, and $h(x,y,t)=s-b$ is ice thickness.

Then the derivatives of function $u(x,y,z,t)$ are

$$\frac{\partial u}{\partial x}=c_x\gamma_1 h^{\gamma_1-1}\frac{\partial h}{\partial x}\left(1-d^{\lambda_1}\right)-c_x\lambda_1 h^{\gamma_1}d^{\lambda_1-1}\frac{\partial d}{\partial x}-\frac{c_{bx}}{h^2}\frac{\partial h}{\partial x}, \tag{65}$$
$$\frac{\partial u}{\partial z}=c_x\lambda_1 h^{\gamma_1-1}d^{\lambda_1-1}.$$

Substituting (64) and (65) into (61) and using relations $\frac{\partial b}{\partial y}\frac{s-z}{s-b}+\frac{\partial s}{\partial y}\frac{z-b}{s-b}=h\frac{\partial d}{\partial x}$ generates a first-order quasi-linear partial differential equation with four independent variables ($x$,$y$, $z$, and $t$) and only one dependent variable ($v$):

$$\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\left(b'_y\frac{s-z}{s-b}+s'_y\frac{z-b}{s-b}\right)+v\frac{s'_y-b'_y}{s-b}$$
$$+c_x(1+\gamma_1)(s'_x-b'_x)(s-b)^{\gamma_1-1}\left[1-\left(\frac{s-z}{s-b}\right)^{\lambda_1}\right]+\frac{1}{s-b}\left(\frac{\partial s}{\partial t}-\dot{a}\right)=0, \tag{66}$$

where $s'_x=\frac{\partial s}{\partial x}, s'_y=\frac{\partial s}{\partial y}, b'_x=\frac{\partial b}{\partial x}, b'_y=\frac{\partial b}{\partial y}$.

The *characteristic system* of Eq. (66) is as follows:

$$\frac{dy}{1}=\frac{dz}{b'_y\frac{s-z}{s-b}+s'_y\frac{z-b}{s-b}} \tag{67}$$
$$=-\frac{dv}{v\frac{s'_y-b'_y}{s-b}+c_x(\gamma_1+1)(s'_x-b'_x)(s-b)^{\gamma_1-1}\left[1-\left(\frac{s-z}{s-b}\right)^{\lambda_1}\right]+\frac{1}{s-b}\left(\frac{\partial s}{\partial t}-\dot{a}\right)}.$$

Two independent particular solutions of this system can be found by solving the equations:

$$\frac{dy}{1}=\frac{dz}{b'_y\frac{s-z}{s-b}+s'_y\frac{z-b}{s-b}}, \tag{68}$$

$$\frac{dy}{1}=-\frac{dv}{v\frac{s'_y-b'_y}{s-b}+c_x(\gamma_1+1)(s'_x-b'_x)(s-b)^{\gamma_1-1}\left[1-\left(\frac{s-z}{s-b}\right)^{\lambda_1}\right]+\frac{1}{s-b}\left(\frac{\partial s}{\partial t}-\dot{a}\right)}. \tag{69}$$

Equation (68) has a solution

$$\frac{z}{s-b}=\frac{b}{s-b}+c_1\text{, or } c_1=\frac{z-b}{s-b}, \tag{70}$$

where $c_1$ is a constant.



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Equation (69) can be re-written as follows:

$$\frac{dv}{dy}+\frac{s'_y-b'_y}{s-b}v=-c_x(\gamma_1+1)(s'_x-b'_x)(s-b)^{\gamma_1-1}\left[1-\left(\frac{s-z}{s-b}\right)^{\lambda_1}\right]-\frac{1}{s-b}\left(\frac{\partial s}{\partial t}-\dot{a}\right). \quad (71)$$

This is a first-order ordinary differential equation. The solution of the homogeneous equation is

$$v=\frac{a(y)}{s-b}. \quad (72)$$

where $a(y)$ is an unknown function.

Substituting (72) into (71), we obtain an equation for $a$:

$$a'(y)=-c_x(\gamma_1+1)(s'_x-b'_x)(s-b)^{\gamma_1}\left[1-\left(\frac{s-z}{s-b}\right)^{\lambda_1}\right]-\left(\frac{\partial s}{\partial t}-\dot{a}\right). \quad (73)$$

Equation (73) has a solution:

$$a(y)=-\int\left\{c_x(\gamma_1+1)(s'_x-b'_x)(s-b)^{\gamma_1}\left[1-\left(\frac{s-z}{s-b}\right)^{\lambda_1}\right]+\left(\frac{\partial s}{\partial t}-\dot{a}\right)\right\}dy+c_2. \quad (74)$$

Substituting (74) into Eq. (72), we obtain

$$v=\frac{-\int\left\{c_x(\gamma_1+1)(s'_x-b'_x)(s-b)^{\gamma_1}\left[1-\left(\frac{s-z}{s-b}\right)^{\lambda_1}\right]+\left(\frac{\partial s}{\partial t}-\dot{a}\right)\right\}dy+c_2}{s-b} \quad (75)$$

or

$$c_2=v(s-b)+\int\left\{c_x(\gamma_1+1)(s'_x-b'_x)(s-b)^{\gamma_1}\left[1-\left(\frac{s-z}{s-b}\right)^{\lambda_1}\right]+\left(\frac{\partial s}{\partial t}-\dot{a}\right)\right\}dy \quad (76)$$

Then, the general solution of Eq. (66) can be written as

$$\theta\left(v(s-b)+\int\left\{c_x(\gamma_1+1)(s'_x-b'_x)(s-b)^{\gamma_1}\left[1-\left(\frac{s-z}{s-b}\right)^{\lambda_1}\right]+\left(\frac{\partial s}{\partial t}-\dot{a}\right)\right\}dy,\frac{z-b}{s-b}\right)=0, \quad (77)$$

where $\theta$ is an arbitrary function of two variables. With Eq. (77) solved for $v$, the general solution can be written in the form

$$v(x,y,z,t) = \frac{1}{s-b}\vartheta\left(\frac{z-b}{s-b}\right) - \frac{1}{s-b}\int\left\{c_x(\gamma_1+1)(s'_x-b'_x)(s-b)^{\gamma_1}\left[1-\left(\frac{s-z}{s-b}\right)^{\lambda_1}\right]+\left(\frac{\partial s}{\partial t}-\dot{a}\right)\right\}dy, \tag{78}$$

where $\vartheta$ is an arbitrary function of one variable.

If we assume again that function $\vartheta$ in (78) is of the form

$$\vartheta(x) = c_y\left[1-(1-x)^{\lambda_2}\right]+c_{by}, \tag{79}$$

where $\lambda_2$, $c_y$, and $c_{by}$ are constants, then functions (63), (59), and (78) satisfying the mass balance equation and the kinematic boundary conditions are as follows:

$$u(x,y,z,t) = c_x(s-b)^{\gamma_1}\left[1-\left(\frac{s-z}{s-b}\right)^{\lambda_1}\right]+c_{bx}\frac{1}{s-b}, \tag{80}$$

$$v(x,y,z,t) = \frac{c_y}{s-b}\left[1-\left(\frac{s-z}{s-b}\right)^{\lambda_2}\right]+c_{by}\frac{1}{s-b} - \frac{1}{s-b}\int c_x\left\{(\gamma_1+1)(s'_x-b'_x)(s-b)^{\gamma_1}\left[1-\left(\frac{s-z}{s-b}\right)^{\lambda_1}\right]+\left(\frac{\partial s}{\partial t}-\dot{a}\right)\right\}dy, \tag{81}$$

$$w(x,y,z,t) = u(x,y,z)\left(\frac{\partial b}{\partial x}\frac{s-z}{s-b}+\frac{\partial s}{\partial x}\frac{z-b}{s-b}\right) + v(x,y,z)\left(\frac{\partial b}{\partial y}\frac{s-z}{s-b}+\frac{\partial s}{\partial y}\frac{z-b}{s-b}\right)+\left(\frac{\partial s}{\partial t}-\dot{a}\right)\frac{z-b}{s-b}, \tag{82}$$

The manufactured solution for the ice pressure can be chosen again as in Pattyn's higher-order model (Pattyn, 2003):

$$\tilde{p} = \sigma'_{\tilde{x}\tilde{x}} + \sigma'_{\tilde{y}\tilde{y}} - \tilde{\rho}\tilde{g}(\tilde{s}-\tilde{z}) = 2\tilde{\mu}\frac{\partial \tilde{u}}{\partial \tilde{x}} + 2\tilde{\mu}\frac{\partial \tilde{v}}{\partial \tilde{y}} - \tilde{\rho}\tilde{g}(\tilde{s}-\tilde{z}),$$

or in nondimensional form:

$$p = 2\mu\frac{\partial u}{\partial x} + 2\mu\frac{\partial v}{\partial y} - (s-z), \tag{83}$$

where non-dimensional ice viscosity

$$\mu = \left[\frac{1}{4}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \frac{1}{4}\left(\frac{1}{\delta}\frac{\partial u}{\partial z} + \delta\frac{\partial w}{\partial x}\right)^2 + \frac{1}{4}\left(\frac{1}{\delta}\frac{\partial v}{\partial z} + \delta\frac{\partial w}{\partial y}\right)^2 \right. \tag{84}$$
$$\left. - \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\frac{\partial w}{\partial z} - \frac{\partial v}{\partial y}\frac{\partial w}{\partial z}\right]^{\frac{1-n}{2n}}.$$

The constructed velocities satisfy the surface and bed kinematic boundary conditions (22–23) and the mass conservation Eq. (17). However, the constructed velocities and pressure do not necessarily satisfy the conservation of momentum equations and the basal and surface boundary conditions. To make the constructed functions into exact solutions of these equations, we substitute them into those equations and calculate the right-hand side functions which accommodate the solutions. This can be done when specific surface $s(x,y,t)$ and bed $b(x,y)$ are chosen.

### 4.1 A time-dependent analytical solution for a flow with a sinusoidal bed

To generate a particular solution, assume a flow with zero accumulation/ablation rate, $\dot{a} = 0$, a sinusoidal bed defined similar to the bed in the benchmark experiment A in (Pattyn et al., 2008), and an ice surface that changes from a linear sloping surface to the one that is draped over the bed:

$$s(x,y,t) = s_0(x) + \eta(x,y)\gamma(t), \; s_0(x) = -x \cdot \tan(\alpha), \tag{85}$$

$$b(x,y) = s_0(x) + \eta(x,y) - 1, \tag{86}$$



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where

$$\eta(x,y) = \frac{1}{2}\sin(2\pi x)\sin(2\pi y),$$
$$\gamma(t) = 1 - e^{-c_t t},\ c_t \text{ is a constant}.$$

To calculate integral in (81), substitute functions (85–86) for bed and surface into the integral in (81). Since it is difficult to calculate the integral analytically for general constants $\gamma_1$ and $\lambda_1$, particular values, for example, $\gamma_1 = 1$ and $\lambda_1 = 1$, can be chosen.

#### 4.1.1 Parameters $\boldsymbol{\gamma_1 = 1, \lambda_1 = 1}$

$$I = \int \left[ 2c_x(z-b)(s'_x - b'_x) + \left( \frac{\partial s}{\partial t} - \dot{a} \right) \right] dy \tag{87}$$

$$= \int \left\{ [z - s_0(x) + 1 - \eta(x,y)]\, 2c_x \eta'_x (\gamma(t) - 1) + \gamma'(t)\eta - \dot{a} \right\} dy$$

$$= 2\pi c_x(\gamma(t)-1)\cos(2\pi x) \int \left[ z - s_0(x) + 1 - \frac{1}{2}\sin(2\pi x)\sin(2\pi y) \right] \sin(2\pi y)\, dy$$

$$+ \frac{\gamma'(t)}{2}\sin(2\pi x) \int \sin(2\pi y)\, dy - \int \dot{a}\, dy$$

$$= 2\pi c_x(\gamma(t)-1)\cos(2\pi x) \left\{ -\frac{z - s_0(x) + 1}{2\pi} - \frac{1}{4}\sin(2\pi x) \int [1 - \cos(4\pi y)]\, dy \right\}$$

$$- \frac{\gamma'(t)}{4\pi}\sin(2\pi x)\cos(2\pi y) - \dot{a}y$$

$$= c_x(1-\gamma(t))\cos(2\pi x)\cos(2\pi y)(z - s_0(x) + 1) - \frac{\gamma'(t)}{4\pi}\sin(2\pi x)\cos(2\pi y)$$

$$+ \left[ \frac{\pi}{4} c_x (1-\gamma(t))\sin(4\pi x) - \dot{a} \right] y - \frac{c_x}{16}(1-\gamma(t))\sin(4\pi x)\sin(4\pi y). \tag{88}$$



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If we substitute the calculated integral and functions (85–86) for bed and surface into (80–82), we obtain the following formulas for velocities:

$$u(x,y,z,t) = c_x(z-b) + c_{bx}\frac{1}{s-b}, \tag{89}$$

$$v(x,y,z,t) = \frac{c_y}{s-b}\left[1-\left(\frac{s-z}{s-b}\right)^{\lambda_2}\right] - \frac{l}{s-b} + c_{by}\frac{1}{s-b}, \tag{90}$$

$$\begin{aligned} w(x,y,z,t) = {} & u(x,y,z,t)\left(\frac{\partial b}{\partial x}\frac{s-z}{s-b} + \frac{\partial s}{\partial x}\frac{z-b}{s-b}\right) \\ & + v(x,y,z,t)\left(\frac{\partial b}{\partial y}\frac{s-z}{s-b} + \frac{\partial s}{\partial y}\frac{z-b}{s-b}\right) + \left(\frac{\partial s}{\partial t} - \dot{a}\right)\frac{z-b}{s-b}. \end{aligned} \tag{91}$$

For a flow down an infinite plane with a mean inclination $\tan(\alpha)$, periodic boundary conditions for a function $f$ are defined as follows: $f(0,y,z+\tan(\alpha)) = f(1,y,z)$, $f(x,0,z+\tan(\alpha)) = f(x,1,z)$.

The constructed solutions (89–91), (83) satisfy periodic boundary conditions only in the horizontal $x$-direction and do not satisfy periodic boundary conditions in the horizontal $y$-direction for all values of the input parameters. To satisfy periodic boundary conditions in all lateral directions, the accumulation-ablation rate may be chosen as follows: $\dot{a} = \dot{a}(x,t) = \frac{\pi c_x(1-\gamma(t))}{4}\sin(4\pi x)$.

Appendix B contains the formulas that can be used to calculate the compensatory stress terms for the momentum equation. For the 3-D ice-stream flow over a bumpy bed experiment, the parameters of the flow are chosen as follows: aspect ratio $\delta = \frac{1}{80}$, the starting linear slope of the ice surface $\alpha = 0.5^\circ$, sliding bed parameters $c_{bx} = c_{by} = 10^{-8}$, and the remaining constants in (89) and (90) $c_x = c_y = 10^{-6}$, $\lambda_2 = 4$, and $c_t = 10^{-6}$. As in 2-D case, all graphs are given for the dimensional values of variables which are calculated from non-dimensional values using formulas (14).

Figure 10 shows the bed (86) and the ice surface (85) at the time zero and at the time when the steady-state solution is reached. Ice flow is from left to right. The ice surface changes from a linear sloping surface to the surface draped over the topography of the bed. Ice thickness is spatially uniform when the steady-state solution is reached.

Figures 11, 12, and 13 show the horizontal and vertical velocity and pressure at the beginning and at the time when the steady-state solution is reached. At the steady-state, the horizontal velocity field is smoothed out, both $x$- and $y$-horizontal velocities are almost spatially uniform (≈46 km/a).

Figure 14 shows the norm of the velocity along the $y=1/4$ slide at the beginning and at the time when the steady-state is reached. Figure 15 shows the transformation over time of the norm of the surface velocity along the $y=1/4$ slide. At the beginning, velocity has two local maximums, over the bump and over the bed where the bed changes the most. At the steady-state position, the norm of the velocity is spatially uniform and at each vertical slide is increasing with ice thickness.

### 4.2 A steady-state analytical solution for a flow with a linear sloping surface and a sinusoidal bed

To generate a steady-state solution, assume that in (85) the function $\gamma(t)=0$, that is, a linear sloping surface and a sloping sinusoidal bed are defined as in the benchmark experiment A in (Pattyn et al., 2008).

$$s(x,y) = -x\cdot\tan(\alpha), \tag{92}$$

$$b(x,y) = s(x,y) - 1 + \frac{1}{2}\sin(2\pi x)\sin(2\pi y). \tag{93}$$

The coefficients are $\alpha=0.5^\circ$, $\lambda_2=2.25$, $c_x=c_y=1$, $c_{bx}=c_{by}=0$, $\delta=\frac{1}{80}$, and accumulation rate $\dot{a}=\frac{\pi\sin(4\pi x)}{4}$.

All functions, the surface horizontal $x$- and $y$-velocities, the vertical $z$-velocities as well as the surface ice pressure, for this steady-state experiment are very similar to the corresponding graphs in Figs. 11, 12, and 13 of the time-dependent experiment at the beginning time.



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## 5 Conclusions

The detailed constructions of exact solutions to 3-D and 2-D flowline time-dependent and steady-state isothermal full-Stokes ice sheet problems are presented. The solutions are valid for non-linear Glen-type flow. The construction of exact solutions done by using manufactured solution technique (Bueler et al., 2007) while the suggested experiments follow directly from ice sheet intercomparison (Pattyn et al., 2008).

The steady-state solutions, constructed in this paper, are variations of the benchmark experiments A and B in (Pattyn et al., 2008). However, by substituting different ice surface and bed geometry formulas into the derived formulas, analytical solutions for different geometries can also be constructed.

Although artificially constructed, the solutions may be useful for testing numerical methods. They offer several benefits to potential ice sheet modelers. By changing a parameter value, the analytical solutions will allow the modelers to investigate their algorithms for a different range of aspect ratios as well as for different, frozen or sliding, basal boundaries. The lateral boundary conditions can be specified as periodic boundary conditions or as essential Dirichlet conditions. Specifying Dirichlet conditions, when the exact solutions are specified as inflow and outflow boundary conditions, allows the modelers to check the model accuracy in the inside of the problem domain.



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## Appendix A

## Calculation of compensatory stress functions in 2-D flowline diagnostic equations

### A1 Compensatory terms in diagnostic equations and in the boundary conditions

The constructed velocities (48–49) satisfy the 2-D versions of the surface and bed kinematic boundary conditions (22–23) and the mass conservation Eq. (17) but do not necessarily satisfy the conservation of momentum Eqs. (18–20) and its basal and surface boundary conditions (24–26) and (27–29). Following (Bueler et al., 2007), we introduce compensatory stresses $\Sigma_x$ and $\Sigma_z$ in the conservation of momentum equations to make the chosen velocity and pressure functions into exact solutions of the equations.

$$\delta \frac{\partial\left(2\mu\frac{\partial u}{\partial x}+p\right)}{\partial x}+\frac{\partial\left(\mu\left(\frac{1}{\delta}\frac{\partial u}{\partial z}+\delta\frac{\partial w}{\partial x}\right)\right)}{\partial z}=\Sigma_x, \tag{A1}$$

$$\delta \frac{\partial\left(\mu\left(\delta\frac{\partial w}{\partial x}+\frac{1}{\delta}\frac{\partial u}{\partial z}\right)\right)}{\partial x}+\frac{\partial\left(2\mu\frac{\partial w}{\partial z}+p\right)}{\partial z}-1=\Sigma_z, \tag{A2}$$

To make the chosen velocities satisfy the boundary conditions, we introduce compensatory terms $\upsilon_x, \upsilon_z, \tau_b$, and $\tau_z$ in the boundary conditions.

At the upper surface $s(x,t)$, the boundary conditions are as follows:

$$\frac{1}{\sqrt{1+\delta^2\left(\frac{ds}{dx}\right)^2}}\left[-\delta\frac{ds}{dx}\left(2\mu\frac{\partial u}{\partial x}+p\right)+\mu\left(\frac{1}{\delta}\frac{\partial u}{\partial z}+\delta\frac{\partial w}{\partial x}\right)\right]=\upsilon_x, \tag{A3}$$

$$\frac{1}{\sqrt{1+\delta^2\left(\frac{ds}{dx}\right)^2}}\left[-\delta\frac{ds}{dx}\left(\mu\left(\delta\frac{\partial w}{\partial x}+\frac{1}{\delta}\frac{\partial u}{\partial z}\right)\right)+\left(2\mu\frac{\partial w}{\partial z}+p\right)\right]=\upsilon_z. \tag{A4}$$

At the lower surface $b(x)$, they are as follows:

$$\frac{1}{\sqrt{1+\delta^2\left(\frac{db}{dx}\right)^2}}\left[\delta\frac{db}{dx}\left(2\mu\frac{\partial u}{\partial x}+p\right)-\mu\left(\frac{1}{\delta}\frac{\partial u}{\partial z}+\delta\frac{\partial w}{\partial x}\right)\right]=\tau_x, \quad \text{(A5)}$$

$$\frac{1}{\sqrt{1+\delta^2\left(\frac{db}{dx}\right)^2}}\left[\delta\frac{db}{dx}\left(\mu\left(\delta\frac{\partial w}{\partial x}+\frac{1}{\delta}\frac{\partial u}{\partial z}\right)\right)-\left(2\mu\frac{\partial w}{\partial z}+p\right)\right]+1=\tau_z. \quad \text{(A6)}$$

## A2 Calculation of derivatives

Calculation of the compensatory stress terms requires calculation of derivatives of the exact solutions (48), (49), and (51). To simplify calculation of the derivatives, we re-write these functions as follows:

$$u(x,z,t)=\frac{1}{h}\left[c_x\left(1-d^\lambda\right)+c_b-\int\left(\frac{\partial s}{\partial t}-\dot{a}\right)dx\right], \quad \text{(A7)}$$

$$w(x,z,t)=uh\frac{\partial d}{\partial x}+\left(\frac{\partial s}{\partial t}-\dot{a}\right)(1-d), \quad \text{(A8)}$$

where $h=h(x,t)$ is ice thickness and $d=d(x,z,t)=\frac{s-z}{s-b}$ is scaled ice depth. Then, the first derivatives of functions (A7–A8) are

$$\begin{aligned}
\frac{\partial u}{\partial z}&=\frac{c_x\lambda}{h^2}d^{\lambda-1},\\
\frac{\partial u}{\partial x}&=-\frac{1}{h}\left[\frac{\partial h}{\partial x}u+c_x\lambda d^{\lambda-1}\frac{\partial d}{\partial x}+\frac{\partial s}{\partial t}-\dot{a}\right]=-\frac{1}{h}\left[\frac{\partial h}{\partial x}u+h^2\frac{\partial u}{\partial z}\frac{\partial d}{\partial x}+\frac{\partial s}{\partial t}-\dot{a}\right],\\
\frac{\partial w}{\partial z}&=-\frac{\partial u}{\partial x}, \qquad \text{(A9)}\\
\frac{\partial w}{\partial x}&=\frac{\partial u}{\partial x}h\frac{\partial d}{\partial x}+u\frac{\partial h}{\partial x}\frac{\partial d}{\partial x}+uh\frac{\partial^2 d}{\partial x^2}+\left(\frac{\partial^2 s}{\partial x\partial t}-\frac{\partial \dot{a}}{\partial x}\right)(1-d)-\left(\frac{\partial s}{\partial t}-\dot{a}\right)\frac{\partial d}{\partial x},
\end{aligned}$$

and the second derivatives are

$$\begin{aligned}
\frac{\partial^2 u}{\partial z^2} &= -\frac{c_x\lambda(\lambda-1)}{h^3}d^{\lambda-2}, \qquad\qquad (A10)\\
\frac{\partial^2 u}{\partial x^2} &= -\frac{1}{h}\left[\frac{\partial^2 h}{\partial x^2}u + 2\frac{\partial h}{\partial x}\frac{\partial u}{\partial x} - h^3\frac{\partial^2 u}{\partial z^2}\left(\frac{\partial d}{\partial x}\right)^2 + h^2\frac{\partial u}{\partial z}\frac{\partial^2 d}{\partial x^2} + \frac{\partial^2 s}{\partial x\partial t} - \frac{\partial \dot{a}}{\partial x}\right],\\
\frac{\partial^2 u}{\partial x\partial z} &= -\frac{2c_x\lambda}{h^3}d^{\lambda-1}\frac{\partial h}{\partial x} + \frac{c_x\lambda(\lambda-1)}{h^2}d^{\lambda-2}\frac{\partial d}{\partial x} = -\frac{1}{h}\left(2\frac{\partial u}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial d}{\partial x}\frac{\partial^2 u}{\partial z^2}\right),\\
\frac{\partial^2 w}{\partial x^2} &= \frac{\partial^2 u}{\partial x^2}h\frac{\partial d}{\partial x} + 2\frac{\partial u}{\partial x}\frac{\partial h}{\partial x}\frac{\partial d}{\partial x} + 2\frac{\partial u}{\partial x}h\frac{\partial^2 d}{\partial x^2} + u\frac{\partial^2 h}{\partial x^2}\frac{\partial d}{\partial x} + 2u\frac{\partial h}{\partial x}\frac{\partial^2 d}{\partial x^2} + uh\frac{\partial^3 d}{\partial x^3}\\
&\quad + \left(\frac{\partial^3 s}{\partial x^2\partial t} - \frac{\partial^2 \dot{a}}{\partial x^2}\right)(1-d) - 2\left(\frac{\partial^2 s}{\partial x\partial t} - \frac{\partial \dot{a}}{\partial x}\right)\frac{\partial d}{\partial x} - \left(\frac{\partial s}{\partial t} - \dot{a}\right)\frac{\partial^2 d}{\partial x^2},\\
\frac{\partial^2 w}{\partial x\partial z} &= -\frac{\partial^2 u}{\partial x^2},\\
\frac{\partial^2 w}{\partial z^2} &= -\frac{\partial^2 u}{\partial x\partial z}.
\end{aligned}$$

where, for a surface (52) and a sinusoidal bed (53),

$$\begin{aligned}
&\frac{\partial h}{\partial x} = \eta'(x)(\gamma(t)-1), \quad \frac{\partial s}{\partial x} = s_0'(x) + \eta'(x)\gamma(t),\\
&\frac{\partial s}{\partial t} = \eta(x)\gamma'(t), \quad \int\frac{\partial s}{\partial t}dx = \gamma'(t)\int\eta(x)dx,\\
&\frac{\partial^2 s}{\partial x\partial t} = \eta'(x)\gamma'(t), \quad \frac{\partial^3 s}{\partial x^2\partial t} = \eta''(x)\gamma'(t),\\
&\frac{\partial^2 h}{\partial x^2} = \eta''(x)(\gamma(t)-1), \quad \frac{\partial^3 h}{\partial x^3} = \eta'''(x)(\gamma(t)-1),
\end{aligned}$$

$$\frac{\partial^2 s}{\partial x^2} = \eta''(x)\gamma(t), \; \frac{\partial^3 s}{\partial x^3} = \eta'''(x)\gamma(t),$$

$$\frac{\partial d}{\partial x} = \frac{1}{h}\left(\frac{\partial s}{\partial x} - \frac{\partial h}{\partial x}d\right), \; \frac{\partial d}{\partial z} = -\frac{1}{h}, \tag{A11}$$

$$\frac{\partial^2 d}{\partial x^2} = \frac{1}{h}\left[\frac{\partial^2 s}{\partial x^2} - 2\frac{\partial h}{\partial x}\frac{\partial d}{\partial x} - \frac{\partial^2 h}{\partial x^2}d\right],$$

$$\frac{\partial^3 d}{\partial x^3} = \frac{1}{h}\left[\frac{\partial^3 s}{\partial x^3} - 3\frac{\partial^2 h}{\partial x^2}\frac{\partial d}{\partial x} - 3\frac{\partial h}{\partial x}\frac{\partial^2 d}{\partial x^2} - d\frac{\partial^3 h}{\partial x^3}\right]$$

If we name the expression

$$\nu = \frac{1}{2}\left(\frac{1}{\delta}\frac{\partial u}{\partial z} + \delta\frac{\partial w}{\partial x}\right)^2 - \frac{\partial u}{\partial x}\frac{\partial w}{\partial z}, \tag{A12}$$

then $\mu = \nu^{\frac{1-n}{2n}}$.

For further calculations we need the following derivatives:

$$\frac{\partial \mu}{\partial x} = \frac{1-n}{2n}\frac{\mu}{\nu}\left[\frac{1}{2}\left(\frac{1}{\delta}\frac{\partial u}{\partial z} + \delta\frac{\partial w}{\partial x}\right)\left(\frac{1}{\delta}\frac{\partial^2 u}{\partial x \partial z} + \delta\frac{\partial^2 w}{\partial x^2}\right) - \frac{\partial^2 u}{\partial x^2}\frac{\partial w}{\partial z} - \frac{\partial u}{\partial x}\frac{\partial^2 w}{\partial x \partial z}\right], \tag{A13}$$

$$\frac{\partial \mu}{\partial z} = \frac{1-n}{2n}\frac{\mu}{\nu}\left[\frac{1}{2}\left(\frac{1}{\delta}\frac{\partial u}{\partial z} + \delta\frac{\partial w}{\partial x}\right)\left(\frac{1}{\delta}\frac{\partial^2 u}{\partial z^2} + \delta\frac{\partial^2 w}{\partial x \partial z}\right) - \frac{\partial^2 u}{\partial x \partial z}\frac{\partial w}{\partial z} - \frac{\partial u}{\partial x}\frac{\partial^2 w}{\partial z^2}\right],$$

Substituting (A9–A10) and (A13–A14) into (A1–A2), (A3–A4), and (A5–A6) generate formulas for compensatory terms $\Sigma_x, \Sigma_z, \upsilon_x, \upsilon_z, \tau_x$, and $\tau_z$.

If constant $\lambda$ in (47) is chosen so that $\lambda > 2$, then the calculation of the second derivatives is well defined.



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## Appendix B

## Calculation of compensatory stress functions in 3-D full-Stokes diagnostic equations

### B1 Compensatory terms in diagnostic equations and in the boundary conditions

The constructed velocities (80)-(82) satisfy the surface and bed kinematic boundary conditions (22) - (23) and the mass conservation Eq. (17). They do not necessarily satisfy the conservation of momentum equations and its basal and surface boundary conditions. Following (Bueler et al., 2007), we introduce compensatory stresses $\Sigma_x$, $\Sigma_y$, and $\Sigma_z$ in the conservation of momentum equations to make the chosen velocity functions into exact solutions of the equations.

$$\delta\frac{\partial\left(2\mu\frac{\partial u}{\partial x}+p\right)}{\partial x}+\delta\frac{\partial\left(\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right)}{\partial y}+\frac{\partial\left(\mu\left(\frac{1}{\delta}\frac{\partial u}{\partial z}+\delta\frac{\partial w}{\partial x}\right)\right)}{\partial z}=\Sigma_x, \tag{B1}$$

$$\delta\frac{\partial\left(\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right)}{\partial x}+\delta\frac{\partial\left(2\mu\frac{\partial v}{\partial y}+p\right)}{\partial y}+\frac{\partial\left(\mu\left(\frac{1}{\delta}\frac{\partial v}{\partial z}+\delta\frac{\partial w}{\partial y}\right)\right)}{\partial z}=\Sigma_y, \tag{B2}$$

$$\delta\frac{\partial\left(\mu\left(\delta\frac{\partial w}{\partial x}+\frac{1}{\delta}\frac{\partial u}{\partial z}\right)\right)}{\partial x}+\delta\frac{\partial\left(\mu\left(\delta\frac{\partial w}{\partial y}+\frac{1}{\delta}\frac{\partial v}{\partial z}\right)\right)}{\partial y}+\frac{\partial\left(2\mu\frac{\partial w}{\partial z}+p\right)}{\partial z}-1=\Sigma_z. \tag{B3}$$

To make the chosen velocities satisfy the boundary conditions, we introduce compensatory terms $\upsilon_x, \upsilon_y, \upsilon_z, \tau_x, \tau_y$, and $\tau_z$ in the boundary conditions.

At the upper surface $s(x,y,t)$, the boundary conditions are as follows:

$$\frac{1}{r_s}\left[-\delta\frac{\partial s}{\partial x}\left(2\mu\frac{\partial u}{\partial x}+p\right)-\delta\frac{\partial s}{\partial y}\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\mu\left(\frac{1}{\delta}\frac{\partial u}{\partial z}+\delta\frac{\partial w}{\partial x}\right)\right]=\upsilon_x, \tag{B4}$$



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$$\frac{1}{r_s}\left[-\delta\frac{\partial s}{\partial x}\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)-\delta\frac{\partial s}{\partial y}\left(2\mu\frac{\partial v}{\partial y}+p\right)+\mu\left(\frac{1}{\delta}\frac{\partial v}{\partial z}+\delta\frac{\partial w}{\partial y}\right)\right]=\upsilon_y, \tag{B5}$$

$$\frac{1}{r_s}\left[-\delta\frac{\partial s}{\partial x}\left(\mu\left(\delta\frac{\partial w}{\partial x}+\frac{1}{\delta}\frac{\partial u}{\partial z}\right)\right)-\delta\frac{\partial s}{\partial y}\left(\mu\left(\delta\frac{\partial w}{\partial y}+\frac{1}{\delta}\frac{\partial v}{\partial z}\right)\right)+\left(2\mu\frac{\partial w}{\partial z}+p\right)\right]=\upsilon_z, \tag{B6}$$

where $r_s=\sqrt{1+\delta^2\left(\frac{\partial s}{\partial x}\right)^2+\delta^2\left(\frac{\partial s}{\partial y}\right)^2}$.

At the lower surface $b(x,y)$, the boundary conditions are as follows:

$$\frac{1}{r_b}\left[\delta\frac{\partial b}{\partial x}\left(2\mu\frac{\partial u}{\partial x}+p\right)+\delta\frac{\partial b}{\partial y}\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)-\mu\left(\frac{1}{\delta}\frac{\partial u}{\partial z}+\delta\frac{\partial w}{\partial x}\right)\right]=\tau_x, \tag{B7}$$

$$\frac{1}{r_b}\left[\delta\frac{\partial b}{\partial x}\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\delta\frac{\partial b}{\partial y}\left(2\mu\frac{\partial v}{\partial y}+p\right)-\mu\left(\frac{1}{\delta}\frac{\partial v}{\partial z}+\delta\frac{\partial w}{\partial y}\right)\right]=\tau_y, \tag{B8}$$

$$\frac{1}{r_b}\left[\delta\frac{\partial b}{\partial x}\left(\mu\left(\delta\frac{\partial w}{\partial x}+\frac{1}{\delta}\frac{\partial u}{\partial z}\right)\right)+\delta\frac{\partial b}{\partial y}\left(\mu\left(\delta\frac{\partial w}{\partial y}+\frac{1}{\delta}\frac{\partial v}{\partial z}\right)\right)-\left(2\mu\frac{\partial w}{\partial z}+p\right)\right]+1=\tau_z, \tag{B9}$$

where $r_b=\sqrt{1+\delta^2\left(\frac{\partial b}{\partial x}\right)^2+\delta^2\left(\frac{\partial b}{\partial y}\right)^2}$.

## B2 Calculation of derivatives

Calculation of the compensatory stress terms requires calculation of derivatives of the exact solutions (80–82), (83). To simplify calculation of the derivatives, we re-write these functions as follows:

$$u(x,y,z,t)=c_x(z-b)+c_{bx}\frac{1}{h}=c_x(1-d)h+c_{bx}\frac{1}{h} \tag{B10}$$

$$v(x,y,z,t)=\frac{c_y}{h}\left(1-d^{\lambda_2}\right)+\frac{c_{by}-l}{h}, \tag{B11}$$

$$w(x,y,z,t)=uh\frac{\partial d}{\partial x}+vh\frac{\partial d}{\partial y}+\left(\frac{\partial h}{\partial t}-\dot{a}\right)(1-d), \tag{B12}$$



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$$p = 2\mu\frac{\partial u}{\partial x} + 2\mu\frac{\partial v}{\partial y} - (s - z), \tag{B13}$$

where $h = h(x,y,t) = s(x,y,t) - b(x,y)$ is ice thickness and $d = d(x,y,z,t) = \frac{s-z}{h}$ is ice scaled ice depth.

The first derivatives of functions (B10–B12) are as follows:

$$\frac{\partial u}{\partial x} = -c_x\frac{\partial b}{\partial x} - \frac{c_{bx}}{h^2}\frac{\partial h}{\partial x}, \quad \frac{\partial u}{\partial y} = -c_x\frac{\partial b}{\partial y} - \frac{c_{bx}}{h^2}\frac{\partial h}{\partial y}, \quad \frac{\partial u}{\partial z} = c_x, \tag{B14}$$

$$\frac{\partial v}{\partial x} = -\frac{1}{h}\left(v\frac{\partial h}{\partial x} + c_y\lambda_2 d^{\lambda_2-1}\frac{\partial d}{\partial x} + \frac{\partial I}{\partial x}\right),$$

$$\frac{\partial v}{\partial y} = -\frac{1}{h}\left(v\frac{\partial h}{\partial y} + c_y\lambda_2 d^{\lambda_2-1}\frac{\partial d}{\partial y} + \frac{\partial I}{\partial y}\right),$$

$$\frac{\partial v}{\partial z} = \frac{c_y\lambda_2}{h^2}d^{\lambda_2-1} - \frac{1}{h}\frac{\partial I}{\partial z},$$

$$\frac{\partial w}{\partial x} = \frac{\partial u}{\partial x}h\frac{\partial d}{\partial x} + u\frac{\partial h}{\partial x}\frac{\partial d}{\partial x} + uh\frac{\partial^2 d}{\partial x^2} + \frac{\partial v}{\partial x}h\frac{\partial d}{\partial y} + v\frac{\partial h}{\partial x}\frac{\partial d}{\partial y} + vh\frac{\partial^2 d}{\partial x\partial y}$$
$$+ \left(\frac{\partial^2 s}{\partial x\partial t} - \frac{\partial \dot{a}}{\partial x}\right)(1-d) - \left(\frac{\partial s}{\partial t} - \dot{a}\right)\frac{\partial d}{\partial x},$$

$$\frac{\partial w}{\partial y} = \frac{\partial u}{\partial y}h\frac{\partial d}{\partial x} + u\frac{\partial h}{\partial y}\frac{\partial d}{\partial x} + uh\frac{\partial^2 d}{\partial x\partial y} + \frac{\partial v}{\partial y}h\frac{\partial d}{\partial y} + v\frac{\partial h}{\partial y}\frac{\partial d}{\partial y} + vh\frac{\partial^2 d}{\partial y^2}$$
$$+ \left(\frac{\partial^2 s}{\partial y\partial t} - \frac{\partial \dot{a}}{\partial y}\right)(1-d) - \left(\frac{\partial s}{\partial t} - \dot{a}\right)\frac{\partial d}{\partial y},$$

$$\frac{\partial w}{\partial z} = \frac{\partial u}{\partial z}h\frac{\partial d}{\partial x} + uh\frac{\partial^2 d}{\partial x\partial z} + \frac{\partial v}{\partial z}h\frac{\partial d}{\partial y} + vh\frac{\partial^2 d}{\partial y\partial z} + \frac{1}{h}\left(\frac{\partial s}{\partial t} - \dot{a}\right),$$

$$\frac{\partial p}{\partial x} = 2\frac{\partial \mu}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x\partial y}\right) - \frac{\partial s}{\partial x},$$

$$\frac{\partial p}{\partial y} = 2\frac{\partial \mu}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+2\mu\left(\frac{\partial^2 u}{\partial x \partial y}+\frac{\partial^2 v}{\partial y^2}\right)-\frac{\partial s}{\partial y},$$

$$\frac{\partial p}{\partial z} = 2\frac{\partial \mu}{\partial z}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+2\mu\left(\frac{\partial^2 u}{\partial x \partial z}+\frac{\partial^2 v}{\partial y \partial z}\right)+1,$$

and the second derivatives are:

$$\frac{\partial^2 u}{\partial x^2} = -c_x\frac{\partial^2 b}{\partial x^2}+2\frac{c_{bx}}{h^3}\left(\frac{\partial h}{\partial x}\right)^2-\frac{c_{bx}}{h^2}\frac{\partial^2 h}{\partial x^2},\quad \frac{\partial^2 u}{\partial y^2} = -c_x\frac{\partial^2 b}{\partial y^2}+2\frac{c_{bx}}{h^3}\left(\frac{\partial h}{\partial y}\right)^2-\frac{c_{bx}}{h^2}\frac{\partial^2 h}{\partial y^2},$$

$$\frac{\partial^2 u}{\partial x \partial y} = -c_x\frac{\partial^2 b}{\partial x \partial y}+2\frac{c_{bx}}{h^3}\frac{\partial h}{\partial x}\frac{\partial h}{\partial y}-\frac{c_{bx}}{h^2}\frac{\partial^2 h}{\partial x \partial y},\quad \frac{\partial^2 u}{\partial z^2}=0,\quad \frac{\partial^2 u}{\partial x \partial z}=0,\quad \frac{\partial^2 u}{\partial y \partial z}=0, \tag{B15}$$

$$\frac{\partial^2 v}{\partial x^2} = -\frac{1}{h}\left(2\frac{\partial v}{\partial x}\frac{\partial h}{\partial x}+v\frac{\partial^2 h}{\partial x^2}+c_y\lambda_2(\lambda_2-1)d^{\lambda_2-2}\left(\frac{\partial d}{\partial x}\right)^2+c_y\lambda_2 d^{\lambda_2-1}\frac{\partial^2 d}{\partial x^2}+\frac{\partial^2 l}{\partial x^2}\right),$$

$$\frac{\partial^2 v}{\partial x \partial y} = -\frac{1}{h}\left(\frac{\partial v}{\partial x}\frac{\partial h}{\partial y}+\frac{\partial v}{\partial y}\frac{\partial h}{\partial x}+v\frac{\partial^2 h}{\partial x \partial y}+c_y\lambda_2(\lambda_2-1)d^{\lambda_2-2}\frac{\partial d}{\partial x}\frac{\partial d}{\partial y}+c_y\lambda_2 d^{\lambda_2-1}\frac{\partial^2 d}{\partial x \partial y}+\frac{\partial^2 l}{\partial x \partial y}\right),$$

$$\frac{\partial^2 v}{\partial y^2} = -\frac{1}{h}\left(2\frac{\partial v}{\partial y}\frac{\partial h}{\partial y}+v\frac{\partial^2 h}{\partial y^2}+c_y\lambda_2(\lambda_2-1)d^{\lambda_2-2}\left(\frac{\partial d}{\partial y}\right)^2+c_y\lambda_2 d^{\lambda_2-1}\frac{\partial^2 d}{\partial y^2}+\frac{\partial^2 l}{\partial y^2}\right),$$

$$\frac{\partial^2 v}{\partial z \partial x} = -\frac{1}{h}\left(\frac{\partial v}{\partial z}\frac{\partial h}{\partial x}-\frac{c_y\lambda_2(\lambda_2-1)}{h}d^{\lambda_2-2}\frac{\partial d}{\partial x}+c_y\lambda_2 d^{\lambda_2-1}\frac{\partial^2 d}{\partial x \partial z}+\frac{\partial^2 l}{\partial x \partial z}\right),$$

$$\frac{\partial^2 v}{\partial z \partial y} = -\frac{1}{h}\left(\frac{\partial v}{\partial z}\frac{\partial h}{\partial y}-\frac{c_y\lambda_2(\lambda_2-1)}{h}d^{\lambda_2-2}\frac{\partial d}{\partial y}+c_y\lambda_2 d^{\lambda_2-1}\frac{\partial^2 d}{\partial y \partial z}+\frac{\partial^2 l}{\partial y \partial z}\right),$$

$$\frac{\partial^2 v}{\partial z^2} = -\frac{c_y\lambda_2(\lambda_2-1)}{h^3}d^{\lambda_2-2},$$

$$\frac{\partial^2 w}{\partial x^2} = 2\frac{\partial h}{\partial x}\frac{\partial d}{\partial x}\frac{\partial u}{\partial x}+2h\frac{\partial^2 d}{\partial x^2}\frac{\partial u}{\partial x}+h\frac{\partial d}{\partial x}\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 h}{\partial x^2}\frac{\partial d}{\partial x}u+2\frac{\partial h}{\partial x}\frac{\partial^2 d}{\partial x^2}u+h\frac{\partial^3 d}{\partial x^3}u$$



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$$
\begin{aligned}
&+2\frac{\partial h}{\partial x}\frac{\partial d}{\partial y}\frac{\partial v}{\partial x}+2h\frac{\partial^2 d}{\partial x\partial y}\frac{\partial v}{\partial x}+h\frac{\partial d}{\partial y}\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 h}{\partial x^2}\frac{\partial d}{\partial y}v+2\frac{\partial h}{\partial x}\frac{\partial^2 d}{\partial x\partial y}v+h\frac{\partial^3 d}{\partial x^2\partial y}v\\
&+\left(\frac{\partial^3 s}{\partial x^2\partial t}-\frac{\partial^2 \dot{a}}{\partial x^2}\right)(1-d)-2\left(\frac{\partial^2 s}{\partial x\partial t}-\frac{\partial \dot{a}}{\partial x}\right)\frac{\partial d}{\partial x}-\left(\frac{\partial s}{\partial t}-\dot{a}\right)\frac{\partial^2 d}{\partial x^2},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 w}{\partial y^2}=&\;2\frac{\partial h}{\partial y}\frac{\partial d}{\partial x}\frac{\partial u}{\partial y}+2h\frac{\partial^2 d}{\partial x\partial y}\frac{\partial u}{\partial y}+h\frac{\partial d}{\partial x}\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 h}{\partial y^2}\frac{\partial d}{\partial x}u+2\frac{\partial h}{\partial y}\frac{\partial^2 d}{\partial x\partial y}u+h\frac{\partial^3 d}{\partial x\partial y^2}u\\
&+2\frac{\partial h}{\partial y}\frac{\partial d}{\partial y}\frac{\partial v}{\partial y}+2h\frac{\partial^2 d}{\partial y^2}\frac{\partial v}{\partial y}+h\frac{\partial d}{\partial y}\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 h}{\partial y^2}\frac{\partial d}{\partial y}v+2\frac{\partial h}{\partial y}\frac{\partial^2 d}{\partial y^2}v+h\frac{\partial^3 d}{\partial y^3}v\\
&+\left(\frac{\partial^3 s}{\partial y^2\partial t}-\frac{\partial^2 \dot{a}}{\partial y^2}\right)(1-d)-2\left(\frac{\partial^2 s}{\partial y\partial t}-\frac{\partial \dot{a}}{\partial y}\right)\frac{\partial d}{\partial y}-\left(\frac{\partial s}{\partial t}-\dot{a}\right)\frac{\partial^2 d}{\partial y^2},
\end{aligned}
$$

$$
\frac{\partial^2 w}{\partial z^2}=2h\frac{\partial^2 d}{\partial x\partial z}\frac{\partial u}{\partial z}+h\frac{\partial d}{\partial x}\frac{\partial^2 u}{\partial z^2}+h\frac{\partial^3 d}{\partial x\partial z^2}u+2h\frac{\partial^2 d}{\partial y\partial z}\frac{\partial v}{\partial z}+h\frac{\partial d}{\partial y}\frac{\partial^2 v}{\partial z^2}+h\frac{\partial^3 d}{\partial y\partial z^2}v,
$$

$$
\begin{aligned}
\frac{\partial^2 w}{\partial x\partial z}=&\;\frac{\partial h}{\partial x}\frac{\partial d}{\partial x}\frac{\partial u}{\partial z}+h\frac{\partial^2 d}{\partial x^2}\frac{\partial u}{\partial z}+h\frac{\partial d}{\partial x}\frac{\partial^2 u}{\partial x\partial z}+\frac{\partial h}{\partial x}\frac{\partial^2 d}{\partial x\partial z}u+h\frac{\partial^3 d}{\partial x^2\partial z}u+h\frac{\partial^2 d}{\partial x\partial z}\frac{\partial u}{\partial x}\\
&+\frac{\partial h}{\partial x}\frac{\partial d}{\partial y}\frac{\partial v}{\partial z}+h\frac{\partial^2 d}{\partial x\partial y}\frac{\partial v}{\partial z}+h\frac{\partial d}{\partial y}\frac{\partial^2 v}{\partial x\partial z}+\frac{\partial h}{\partial x}\frac{\partial^2 d}{\partial y\partial z}v+h\frac{\partial^3 d}{\partial x\partial y\partial z}v+h\frac{\partial^2 d}{\partial y\partial z}\frac{\partial v}{\partial x}\\
&+\frac{1}{h}\left(\frac{\partial^2 s}{\partial x\partial t}-\frac{\partial \dot{a}}{\partial x}\right)-\frac{1}{h^2}\frac{\partial h}{\partial x}\left(\frac{\partial s}{\partial t}-\dot{a}\right),
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 w}{\partial y\partial z}+&\;\frac{\partial h}{\partial y}\frac{\partial d}{\partial x}\frac{\partial u}{\partial z}+h\frac{\partial^2 d}{\partial x\partial y}\frac{\partial u}{\partial z}+h\frac{\partial d}{\partial x}\frac{\partial^2 u}{\partial y\partial z}+\frac{\partial h}{\partial y}\frac{\partial^2 d}{\partial x\partial z}u+h\frac{\partial^3 d}{\partial x\partial y\partial z}u+h\frac{\partial^2 d}{\partial x\partial z}\frac{\partial u}{\partial y}\\
&+\frac{\partial h}{\partial y}\frac{\partial d}{\partial y}\frac{\partial v}{\partial z}+h\frac{\partial^2 d}{\partial y^2}\frac{\partial v}{\partial z}+h\frac{\partial d}{\partial y}\frac{\partial^2 v}{\partial y\partial z}+\frac{\partial h}{\partial y}\frac{\partial^2 d}{\partial y\partial z}v+h\frac{\partial^3 d}{\partial^2 y\partial z}v+h\frac{\partial^2 d}{\partial y\partial z}\frac{\partial v}{\partial y}\\
&+\frac{1}{h}\left(\frac{\partial^2 s}{\partial y\partial t}-\frac{\partial \dot{a}}{\partial y}\right)-\frac{1}{h^2}\frac{\partial h}{\partial y}\left(\frac{\partial s}{\partial t}-\dot{a}\right),
\end{aligned}
$$

where

$$\frac{\partial s}{\partial x}=\frac{\partial s_0}{\partial x}+\frac{\partial \eta}{\partial x}\gamma(t),\ \frac{\partial s}{\partial y}=\frac{\partial \eta}{\partial y}\gamma(t), \tag{B16}$$

$$\frac{\partial^2 s}{\partial x^2}=\frac{\partial^2 \eta}{\partial x^2}\gamma(t),\ \frac{\partial^2 s}{\partial y^2}=\frac{\partial^2 \eta}{\partial y^2}\gamma(t),\ \frac{\partial^2 s}{\partial x\partial y}=\frac{\partial^2 \eta}{\partial x\partial y}\gamma(t),$$

$$\frac{\partial^2 s}{\partial x\partial t}=\frac{\partial \eta}{\partial x}\frac{\partial \gamma(t)}{\partial t},\ \frac{\partial^2 s}{\partial y\partial t}=\frac{\partial \eta}{\partial y}\frac{\partial \gamma(t)}{\partial t},$$

$$\frac{\partial^3 s}{\partial x^2\partial t}=\frac{\partial^2 \eta}{\partial x^2}\frac{\partial \gamma(t)}{\partial t},\ \frac{\partial^3 s}{\partial y^2\partial t}=\frac{\partial^2 \eta}{\partial y^2}\frac{\partial \gamma(t)}{\partial t},$$

$$h=1+\eta(x,y)(\gamma(t)-1),$$

$$\frac{\partial h}{\partial x}=\frac{\partial \eta}{\partial x}(\gamma(t)-1),\ \frac{\partial h}{\partial y}=\frac{\partial \eta}{\partial y}(\gamma(t)-1),$$

$$\frac{\partial^2 h}{\partial x^2}=\frac{\partial^2 \eta}{\partial x^2}(\gamma(t)-1),\ \frac{\partial^2 h}{\partial y^2}=\frac{\partial^2 \eta}{\partial y^2}(\gamma(t)-1),\ \frac{\partial^2 h}{\partial x\partial y}=\frac{\partial^2 \eta}{\partial x\partial y}(\gamma(t)-1),$$

$$\frac{\partial h}{\partial t}=\eta(x,y)\frac{\partial \gamma(t)}{\partial t},\ \frac{\partial^2 h}{\partial x\partial t}=\frac{\partial \eta}{\partial x}\frac{\partial \gamma(t)}{\partial t},\ \frac{\partial^2 h}{\partial y\partial t}=\frac{\partial \eta}{\partial y}\frac{\partial \gamma(t)}{\partial t},$$

$$\frac{\partial d}{\partial x}=\frac{1}{h}\left(\frac{\partial s}{\partial x}-d\frac{\partial h}{\partial x}\right),\ \frac{\partial d}{\partial y}=\frac{1}{h}\left(\frac{\partial s}{\partial y}-d\frac{\partial h}{\partial y}\right),\ \frac{\partial d}{\partial z}=-\frac{1}{h},$$

$$\frac{\partial^2 d}{\partial x^2}=\frac{1}{h}\left(\frac{\partial^2 s}{\partial x^2}-d\frac{\partial^2 h}{\partial x^2}-2\frac{\partial h}{\partial x}\frac{\partial d}{\partial x}\right),\ \frac{\partial^2 d}{\partial y^2}=\frac{1}{h}\left(\frac{\partial^2 s}{\partial y^2}-d\frac{\partial^2 h}{\partial y^2}-2\frac{\partial h}{\partial y}\frac{\partial d}{\partial y}\right),$$

$$\frac{\partial^2 d}{\partial x\partial y}=\frac{1}{h}\left(\frac{\partial^2 s}{\partial x\partial y}-\frac{\partial h}{\partial x}\frac{\partial d}{\partial y}-\frac{\partial h}{\partial y}\frac{\partial d}{\partial x}-d\frac{\partial^2 h}{\partial x\partial y}\right),\ \frac{\partial^2 d}{\partial x\partial z}=\frac{1}{h^2}\frac{\partial h}{\partial x},\ \frac{\partial^2 d}{\partial y\partial z}=\frac{1}{h^2}\frac{\partial h}{\partial y},$$

$$\frac{\partial^3 d}{\partial x^3}=-\frac{1}{h}\left(3\frac{\partial h}{\partial x}\frac{\partial^2 d}{\partial x^2}+3\frac{\partial^2 h}{\partial x^2}\frac{\partial d}{\partial x}+d\frac{\partial^3 h}{\partial x^3}-\frac{\partial^3 s}{\partial x^3}\right),$$

$$\frac{\partial^3 d}{\partial y^3}=-\frac{1}{h}\left(3\frac{\partial h}{\partial y}\frac{\partial^2 d}{\partial y^2}+3\frac{\partial^2 h}{\partial y^2}\frac{\partial d}{\partial y}+d\frac{\partial^3 h}{\partial y^3}-\frac{\partial^3 s}{\partial y^3}\right),$$

$$\frac{\partial^3 d}{\partial x^2 \partial y} = -\frac{1}{h}\left(\frac{\partial h}{\partial y}\frac{\partial^2 d}{\partial x^2} + 2\frac{\partial h}{\partial x}\frac{\partial^2 d}{\partial x \partial y} + 2\frac{\partial^2 h}{\partial x \partial y}\frac{\partial d}{\partial x} + \frac{\partial^2 h}{\partial x^2}\frac{\partial d}{\partial y} + d\frac{\partial^3 h}{\partial x^2 \partial y} - \frac{\partial^3 s}{\partial x^2 \partial y}\right),$$

$$\frac{\partial^3 d}{\partial x \partial y^2} = -\frac{1}{h}\left(\frac{\partial h}{\partial x}\frac{\partial^2 d}{\partial y^2} + 2\frac{\partial h}{\partial y}\frac{\partial^2 d}{\partial x \partial y} + 2\frac{\partial^2 h}{\partial x \partial y}\frac{\partial d}{\partial y} + \frac{\partial^2 h}{\partial y^2}\frac{\partial d}{\partial x} + d\frac{\partial^3 h}{\partial y^2 \partial x} - \frac{\partial^3 s}{\partial y^2 \partial x}\right),$$

$$\frac{\partial^3 d}{\partial x^2 \partial z} = \frac{1}{h^2}\left[\frac{\partial^2 h}{\partial x^2} - \frac{2}{h}\left(\frac{\partial h}{\partial x}\right)^2\right], \quad \frac{\partial^3 d}{\partial y^2 \partial z} = \frac{1}{h^2}\left[\frac{\partial^2 h}{\partial y^2} - \frac{2}{h}\left(\frac{\partial h}{\partial y}\right)^2\right],$$

$$\frac{\partial^3 d}{\partial x \partial y \partial z} = \frac{1}{h^2}\left[\frac{\partial^2 h}{\partial x \partial y} - \frac{2}{h}\frac{\partial h}{\partial x}\frac{\partial h}{\partial y}\right], \quad \frac{\partial^3 d}{\partial x \partial z^2} = 0, \quad \frac{\partial^3 d}{\partial y \partial z^2} = 0,$$

$$\begin{aligned}\frac{\partial I}{\partial x} &= -c_x(1-\gamma(t))\cos(2\pi y)\left[2\pi(z - s_0(x) + 1)\sin(2\pi x) + s_0'(x)\cos(2\pi x)\right] - \frac{\gamma'(t)}{2}\cos(2\pi x) \\ &+ \cos(2\pi y)\left[\pi^2 c_x(1-\gamma(t))\cos(4\pi x) - \frac{\partial \dot{a}}{\partial x}\right] y - \frac{\pi c_x}{4}(1-\gamma(t))\cos(4\pi x)\sin(4\pi y),\end{aligned}$$

$$\begin{aligned}\frac{\partial I}{\partial y} &= -2\pi c_x(1-\gamma(t))\cos(2\pi x)\sin(2\pi y)(z - s_0(x) + 1) + \frac{\gamma'(t)}{2}\sin(2\pi x)\sin(2\pi y) \\ &+ \left[\frac{\pi}{4}c_x(1-\gamma(t))\sin(4\pi x) - \dot{a}\right] - \frac{\pi c_x}{4}(1-\gamma(t))\sin(4\pi x)\cos(4\pi y),\end{aligned}$$

$$\frac{\partial I}{\partial z} = c_x(1-\gamma(t))\cos(2\pi x)\cos(2\pi y),$$

$$\begin{aligned}\frac{\partial^2 I}{\partial x^2} &= -c_x(1-\gamma(t))\cos(2\pi y)\left[4\pi^2(z - s_0(x) + 1)\cos(2\pi x) - 4\pi s_0'(x)\sin(2\pi x)\right] + \pi\gamma'(t)\sin(2\pi x) \\ &- \cos(2\pi y)\left[4\pi^3 c_x(1-\gamma(t))\sin(4\pi x) + \frac{\partial^2 \dot{a}}{\partial x^2}\right] y + \pi^2 c_x(1-\gamma(t))\sin(4\pi x)\sin(4\pi y),\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 I}{\partial x \partial y} &= 2\pi c_x(1-\gamma(t))\sin(2\pi y)\left[2\pi(z - s_0(x) + 1)\sin(2\pi x) + s_0'(x)\cos(2\pi x)\right] + \pi\gamma'(t)\cos(2\pi x) \\ &+ \sin(2\pi y)\left[\pi^2 c_x(1-\gamma(t))\cos(4\pi x) - \frac{\partial \dot{a}}{\partial x}\right] - \pi^2 c_x(1-\gamma(t))\cos(4\pi x)\cos(4\pi y),\end{aligned}$$



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$$\frac{\partial^2 I}{\partial x \partial z} = -2\pi c_x (1-\gamma(t))\cos(2\pi y)\sin(2\pi x),$$

$$\frac{\partial^2 I}{\partial y^2} = -4\pi^2 c_x (1-\gamma(t))\cos(2\pi x)\cos(2\pi y)(z - s_0(x) + 1) + \pi\gamma'(t)\sin(2\pi x)\cos(2\pi y)$$

$$+ \pi^2 c_x (1-\gamma(t))\sin(4\pi x)\sin(4\pi y),$$

$$\frac{\partial^2 I}{\partial y \partial z} = -2\pi c_x (1-\gamma(t))\cos(2\pi x)\sin(2\pi y).$$

If we name the expression

$$\nu = \frac{1}{4}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \frac{1}{4}\left(\frac{1}{\delta}\frac{\partial u}{\partial z} + \delta\frac{\partial w}{\partial x}\right)^2 + \frac{1}{4}\left(\frac{1}{\delta}\frac{\partial v}{\partial z} + \delta\frac{\partial w}{\partial y}\right)^2 - \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\frac{\partial w}{\partial z} - \frac{\partial v}{\partial y}\frac{\partial w}{\partial z},$$

then $\mu = \nu^{\frac{1-n}{2n}}$.

For further calculations we need the following derivatives:

$$\frac{\partial \mu}{\partial x} = \frac{1-n}{2n}\frac{\mu}{\nu}\left[\frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x^2}\right) + \frac{1}{2}\left(\frac{1}{\delta}\frac{\partial u}{\partial z} + \delta\frac{\partial w}{\partial x}\right)\left(\frac{1}{\delta}\frac{\partial^2 u}{\partial x \partial z} + \delta\frac{\partial^2 w}{\partial x^2}\right)\right.$$

$$+ \frac{1}{2}\left(\frac{1}{\delta}\frac{\partial v}{\partial z} + \delta\frac{\partial w}{\partial y}\right)\left(\frac{1}{\delta}\frac{\partial^2 v}{\partial x \partial z} + \delta\frac{\partial^2 w}{\partial x \partial y}\right)$$

$$\left. - \frac{\partial^2 u}{\partial x^2}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 u}{\partial x^2}\frac{\partial w}{\partial z} - \frac{\partial u}{\partial x}\frac{\partial^2 w}{\partial x \partial z} - \frac{\partial^2 v}{\partial x \partial y}\frac{\partial w}{\partial z} - \frac{\partial v}{\partial y}\frac{\partial^2 w}{\partial x \partial z}\right],$$

$$\frac{\partial \mu}{\partial y} = \frac{1-n}{2n}\frac{\mu}{\nu}\left[\frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y}\right) + \frac{1}{2}\left(\frac{1}{\delta}\frac{\partial u}{\partial z} + \delta\frac{\partial w}{\partial x}\right)\left(\frac{1}{\delta}\frac{\partial^2 u}{\partial y \partial z} + \delta\frac{\partial^2 w}{\partial x \partial y}\right)\right.$$

$$+ \frac{1}{2}\left(\frac{1}{\delta}\frac{\partial v}{\partial z} + \delta\frac{\partial w}{\partial y}\right)\left(\frac{1}{\delta}\frac{\partial^2 v}{\partial y \partial z} + \delta\frac{\partial^2 w}{\partial y^2}\right) \quad \text{(B17)}$$

$$\left. - \frac{\partial^2 u}{\partial x \partial y}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\frac{\partial^2 v}{\partial y^2} - \frac{\partial^2 u}{\partial x \partial y}\frac{\partial w}{\partial z} - \frac{\partial u}{\partial x}\frac{\partial^2 w}{\partial y \partial z} - \frac{\partial^2 v}{\partial y^2}\frac{\partial w}{\partial z} - \frac{\partial v}{\partial y}\frac{\partial^2 w}{\partial y \partial z}\right],$$



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$$\frac{\partial\mu}{\partial z}=\frac{1-n}{2n}\frac{\mu}{v}\left[\frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\left(\frac{\partial^2 u}{\partial y\partial z}+\frac{\partial^2 v}{\partial x\partial z}\right)+\frac{1}{2}\left(\frac{1}{\delta}\frac{\partial u}{\partial z}+\delta\frac{\partial w}{\partial x}\right)\left(\frac{1}{\delta}\frac{\partial^2 u}{\partial z^2}+\delta\frac{\partial^2 w}{\partial x\partial z}\right)\right.$$
$$+\frac{1}{2}\left(\frac{1}{\delta}\frac{\partial v}{\partial z}+\delta\frac{\partial w}{\partial y}\right)\left(\frac{1}{\delta}\frac{\partial^2 v}{\partial z^2}+\delta\frac{\partial^2 w}{\partial y\partial z}\right)$$
$$\left.-\frac{\partial^2 u}{\partial x\partial z}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial x}\frac{\partial^2 v}{\partial y\partial z}-\frac{\partial^2 u}{\partial x\partial z}\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}\frac{\partial^2 w}{\partial z^2}-\frac{\partial^2 v}{\partial y\partial z}\frac{\partial w}{\partial z}-\frac{\partial v}{\partial y}\frac{\partial^2 w}{\partial z^2}\right].$$

Substituting (B10–B14) and (B17) into (B1–B3), (B4–B6), and (B7–B9) generate formulas for compensatory terms $\Sigma_x, \Sigma_y, \Sigma_z, \upsilon_x, \upsilon_y, \upsilon_z, \tau_x, \tau_y$, and $\tau_z$.

If constant $\lambda_2$ in (B11) is chosen so that $\lambda_2 > 2$, then calculation of the velocities' first and second derivatives is well defined.

## Appendix C

## FORTRAN 77 program to calculate exact solutions and compensatory stress functions for 2-D flowline benchmark experiment

### C1 parameter2d.h

```
      LOGICAL steady, dimensional

      DOUBLE PRECISION lambda, Lx
      PARAMETER (lambda=4.,cx=1.e-6,cb=1.e-6,ct=1.e-6)
      PARAMETER (steady=.false., dimensional=.true.)
      PARAMETER(Lx=80000.,nx=100,nz=100,nt=9,dt=1.e+7)
C-------------------------------------------------------------------
C       lambda, cx, cb ,ct - parameters of the manufactured solution;
C       Lx - horizontal length of the domain for dimensional problem;
C       nx,nz - grid sizes, nt - number of time steps, dt - time step;
C       steady = .true. for steady-state experiment,
C              = .false. for time-dependent experiment;
C       dimensional = .true. to dump dimensional values of solutions,
C                   = .false. to dump non-dimensional values of solutions.
C-------------------------------------------------------------------
```



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## C2 exact2d.f

```
C       2-D full-Stokes flowline ice sheet model.

C
C      Parameter of the model: lambda, cx, cb, ct,
C      Parameter of the flow Lx(horizontal length scale of the domain), and
C      Parameter of the grid: nx, nz, nt, dt
C                - assigned in "parameter2d.h".
C      Bed and surface topography defined in SUBROUTINE 'testB'.
C
C       Exact solutions are dumped (in format: x z f) to files:
C          2du       - horizontal velocity u (in km/a, if dimensional),
C          2dw       - vertical velocity w (in km/a, if dimensional),
C          2dp       - pressure p (in kPa, if dimensional),
C          2dunorm   - sqrt(u*u+w*w) (in km/a, if dimensional),
C          2dmu      - effective viscosity (in Pa sec^2, if dimensional).
C       Compensatory stress terms are dumped to files:
C          2dsigx    - horizontal component of momentum equation
C          2dsigz    - vertical component of momentum equation
C          2dsurfx   - horizontal component of top boundary condition
C          2dsurfz   - vertical component of top boundary condition
C          2dbedx    - horizontal component of bottom boundary condition
C          2dbedz    - vertical component of bottom boundary condition
C----------------------------------------------------------------------
      IMPLICIT REAL*8(a-h,o-z)
      include "parameter2d.h"
      DIMENSION u(nx,nz),w(nx,nz),p(nx,nz)
      DIMENSION x(nx),z(nx,nz),bed(nx),surf(nx)
      DIMENSION sigx(nx,nz),sigz(nx,nz)
      DIMENSION bedx(nx),bedz(nx),surfx(nx),surfz(nx)
      DOUBLE PRECISION mu(nx,nz)
      PARAMETER(eps=1.e-20)
C ---
      CALL scales(delta,sx,sz,sp,su,sw,smu,st,ssig,ssigb)
      print *,'steady=',steady,' dimensional=',dimensional
      print *,'model parameters: cx=',cx,' cb=',cb,' ct=',ct
      print *,'domain: ',sx,' by',sz
C ---
C --- beginning of the time loop
C ---
      DO k=1,nt
```

```
      tt=(k-1)*dt
      dx=1./(nx-1)
C ---
C --- beginning of the x- loop
C ---
      DO i=1,nx
        xx=(i-1)*dx
        CALL testB(xx,tt,b,dbdx,d2bdx2,d3bdx3,s,dsdx,
     &              d3sdx3,dsdt,d2sdxdt,d3sdx2dt,dsdtint)
        x(i)=xx
        surf(i)=s
        bed(i)=b
        h=surf(i)-bed(i)
        dhdx=dsdx-dbdx
        d2hdx2=d2sdx2-d2bdx2
        d3hdx3=d3sdx3-d3bdx3
        dz=h/(nz-1)
C ---
C --- beginning of the z- loop
C ---
        DO j=1,nz
          z(i,j)=bed(i)+(j-1)*dz
          d=(surf(i)-z(i,j))/h
          dddz=-1./h
          dddx=(dsdx-dhdx*d)/h
          d2ddx2=(d2sdx2-d2hdx2*d-2.*dhdx*dddx)/h
          d3ddx3=(d3sdx3-3.*d2hdx2*dddx-3.*dhdx*d2ddx2-d*d3hdx3)/h
C ---
C --- Calculate velocities
C ---
          u(i,j)=(cx*(1-d**lambda)+cb-dsdtint)/h
          w(i,j)=u(i,j)*h*dddx+dsdt*(1-d)
C ---
C --- Calculate the first derivatives
C ---
          dudz=cx*lambda*d**(lambda-1)/h**2
          dudx=-( dhdx*u(i,j)+h**2*dudz*dddx+dsdt )/h
          dwdz=-dudx
          dwdx=dudx*h*dddx+u(i,j)*dhdx*dddx
     &    +u(i,j)*h*d2ddx2+d2sdxdt*(1.-d)-dsdt*dddx
C ---
C --- Calculate the second derivatives
```



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```
C ---
            d2udz2=-cx*lambda*(lambda-1)*d**(lambda-2)/h**3
            d2udx2=-( d2hdx2*u(i,j)+2*dhdx*dudx
     &       -h**3*d2udz2*dddx**2+h**2*dudz*d2ddx2-d2sdxdt )/h
            d2udxdz=-( 2*dudx*dudz+dddx*d2udz2 )/h
            d2wdx2=d2udx2*h*dddx + 2*dudx*(dhdx*dddx+h*d2ddx2)
     &     + u(i,j)*(d2hdx2*dddx+2.*dhdx*d2ddx2+h*d3ddx3)
     &     + d3sdx2dt*(1.-d)-2.*d2sdxdt*dddx-dsdt*d2ddx2
            d2wdxdz=-d2udx2
            d2wdz2=-d2udxdz
C ---
C --- Calculate the effective viscosity and pressure:
C ---
            sxz=dudz/delta+delta*dwdx
            dsxzdx=d2udxdz/delta+delta*d2wdx2
            dsxzdz=d2udz2/delta+delta*d2wdxdz
            xnu=0.25*sxz**2-dudx*dwdz
            mu(i,j)=(xnu+eps)**(-1./3.)
            p(i,j)=2.*mu(i,j)*dudx-surf(i)+z(i,j)
C ---
C --- Calculate the compensatory stress terms:
C ---
            dmudx=(-1./3)*mu(i,j)/(xnu+eps)*
     &            (0.5*sxz*dsxzdx - d2udx2*dwdz-dudx*d2wdxdz)
            dmudz=(-1./3)*mu(i,j)/(xnu+eps)*
     &            (0.5*sxz*dsxzdz - d2udxdz*dwdz-dudx*d2wdz2)
            dpdx=2*dmudx*dudx + 2*mu(i,j)*d2udx2-dsdx
            dpdz=2*dmudz*dudx + 2*mu(i,j)*d2udxdz+1
            sigxx=delta* (2*dmudx*dudx+2*mu(i,j)*d2udx2 + dpdx)
            sigxz=dmudz*sxz+mu(i,j)*dsxzdz
            sigx(i,j)=sigxx+sigxz
            sigzx=delta*(dmudx*sxz+mu(i,j)*dsxzdx)
            sigzz=2*dmudz*dwdz+2*mu(i,j)*d2wdz2 + dpdz -1.
            sigz(i,j)=sigzx+sigzz
C ---
C --- Calculate the compensatory stress terms for
C ___       the upper surface boundary conditions:
C ---
          IF(j.eq.nz) THEN
            r=delta*dsdx
            rinv=1./sqrt(1.+r*r)
            sxx=2.*mu(i,j)*dudx+p(i,j)
```



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```
          szz=2.*mu(i,j)*dwdz+p(i,j)
          surfx(i)=rinv*( -r*sxx+mu(i,j)*sxz)
          surfz(i)=rinv*( -r*mu(i,j)*sxz+szz)
        ENDIF
C ---
C --- Calculate the compensatory stress terms for
C ___       the basal boundary conditions:
C ---
        IF(j.eq.1) THEN
          r=delta*dbdx
          rinv=1./sqrt(1.+r*r)
          sxx=2.*mu(i,j)*dudx+p(i,j)
          szz=2.*mu(i,j)*dwdz+p(i,j)
          bedx(i)=rinv*( r*sxx-mu(i,j)*sxz)
          bedz(i)=rinv*( r*mu(i,j)*sxz-szz)+1.
        ENDIF
      ENDDO     !  end of z- loop
      ENDDO     !  end of x- loop
C............................................
      CALL dumpsurf(k,x,z,u,w,surf,bed,
   &            surfx,surfz,bedx,bedz)
C ---
C --- Dump the values at the desired time:
C ---
      IF(k.eq.1 .or. k.eq.nt) THEN
        CALL dump(k,x,z,u,w,p,mu,sigx,sigz,
   &            surfx,surfz,bedx,bedz)
      ENDIF
     ENDDO     !  end of time loop
     STOP
     END
C-------------------------------------------------------------
     SUBROUTINE testB(x,t,b,dbdx,d2bdx2,d3bdx3,s,dsdx,
   &          d3sdx3,dsdt,d2sdxdt,d3sdx2dt,dsdtint)
C-------------------------------------------------------------
C     Return the values of the bed, surface, and their derivatives
C           for a given values of x and t.
C
C     Input:
C       x   - horizontal coordinat of the column of ice,
C       t   - time.
C-------------------------------------------------------------
```

```
      IMPLICIT REAL*8(a-h,o-z)
      include "parameter2d.h"
        tana=tan(0.5)
        omega=8.*atan(1.)   !  2*pi
C ---
        s0x    = -tana*x
        s0xp   = -tana
        eta    = 0.5*sin(omega*x)
        etap   = 0.5*omega*cos(omega*x)
        etap2  = -0.5*omega**2*sin(omega*x)
        etap3  = -0.5*omega**3*cos(omega*x)
        etaint = -0.5/omega*cos(omega*x)
        IF(steady .or. t.eq.0.) THEN
          gamma = 0.
          gammap = 0.
        ELSE
          gamma  = 1-exp(-ct*t)
          gammap = ct*exp(-t)
        ENDIF
C ---
        b         = s0x-1+eta
        s         = s0x+eta*gamma
        dbdx      = s0xp+etap
        d2bdx2    = etap2
        d3bdx3    = etap3
        dsdx      = s0xp+etap*gamma
        d2sdx2    = etap2*gamma
        d3sdx3    = etap3*gamma
        dsdt      = eta*gammap
        dsdtint   = gammap*etaint
        d2sdxdt   = etap*gammap
        d3sdx2dt  = etap2*gammap
      RETURN
      END
C-----------------------------------------------------------------------
      SUBROUTINE dump(k,x,z,u,w,p,mu,sigx,sigz,
     &              surfx,surfz,bedx,bedz)
C-----------------------------------------------------------------------
C     Dumps values of exact solutions and compensatory stresses
C          at the specified time step k.
C
C     Dumps dimensional values if 'dimensional'=.true.
```

```
C           and nondimensional, otherwise.
C----------------------------------------------------------------------
      IMPLICIT REAL*8(a-h,o-z)
      INCLUDE "parameter2d.h"
      DOUBLE PRECISION mu(nx,nz)
      DIMENSION u(nx,nz),w(nx,nz),p(nx,nz)
      DIMENSION x(nx),z(nx,nz),bed(nx),surf(nx)
      DIMENSION sigx(nx,nz),sigz(nx,nz)
      DIMENSION bedx(nx),bedz(nx),surfx(nx),surfz(nx)
      CHARACTER*20 string, out(11)
      CALL scales(delta,sx,sz,sp,su,sw,smu,st,ssig,ssigb)
        IF(k.lt.10) THEN
          WRITE(unit=string,fmt='(I1)') k
        else
          WRITE(unit=string,fmt='(I2)') k
        ENDIF
        out(1)='2du'//string
        out(2)='2dw'//string
        out(3)='2dp'//string
        out(4)='2dunorm'//string
        out(5)='2dmu'//string
        out(6)='2dsigx'//string
        out(7)='2dsigz'//string
        out(8)='2dsigsurfx'//string
        out(9)='2dsigsurfz'//string
        out(10)='2dsigbedx'//string
        out(11)='2dsigbedz'//string
      DO i=21,31
        OPEN(i,file=out(i-20))
      ENDDO
        DO j=1,nz
        DO i=1,nx
          xx=x(i)
          zz=z(i,j)
          WRITE(21,*) xx,zz,u(i,j)*su
          WRITE(22,*) xx,zz,w(i,j)*sw
          WRITE(23,*) xx,zz,p(i,j)*sp
          WRITE(24,*) xx,zz,sqrt(u(i,j)*u(i,j)*su*su
     &                           +w(i,j)*w(i,j)*sw*sw)
          WRITE(25,*) xx,zz,mu(i,j)*smu
          WRITE(26,*) xx,zz,sigx(i,j)*ssig
          WRITE(27,*) xx,zz,sigz(i,j)*ssig
```



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```
      ENDDO
      ENDDO
      DO i=1,nx
        xx=x(i)
        WRITE(28,*) xx,surfx(i)*ssigb
        WRITE(29,*) xx,surfz(i)*ssigb
        WRITE(30,*) xx,bedx(i)*ssigb
        WRITE(31,*) xx,bedz(i)*ssigb
      ENDDO
      DO i=21,31
        CLOSE(i)
      ENDDO
    END
C----------------------------------------------------------------------
      SUBROUTINE dumpsurf(k,x,z,u,w,surf,bed,
     &              surfx,surfz,bedx,bedz)
C----------------------------------------------------------------------
C     Dumps values of bed and surface, as well as compensatory
C       stresses at the top and the bottom surfaces.
C       at the specified time step k.
C     Dumps dimensional values if 'dimensional'=.true.
C           and nondimensional, otherwise.
C----------------------------------------------------------------------
      IMPLICIT REAL*8(a-h,o-z)
      INCLUDE "parameter2d.h"
      DIMENSION x(nx),z(nx,nz),bed(nx),surf(nx)
      DIMENSION u(nx,nz),w(nx,nz)
      DIMENSION bedx(nx),bedz(nx),surfx(nx),surfz(nx)
      CALL scales(delta,sx,sz,sp,su,sw,smu,st,ssig,ssigb)
      IF(k.eq.1) THEN
        OPEN(41,file='2dsb.data')
        OPEN(42,file='2dsurfvel.data')
        OPEN(43,file='2dsurfx.data')
        OPEN(44,file='2dsurfz.data')
        OPEN(45,file='2dbedx.data')
        OPEN(46,file='2dbedz.data')
        DO i=1,nx
          WRITE(41,*) x(i),bed(i)*sz
        ENDDO
          WRITE(41,*) -99999,0
          WRITE(41,*) 'bed'
      ENDIF
```

```
      DO i=1,nx
        xx=x(i)
        WRITE(41,*) xx,surf(i)*sz
        WRITE(42,*) xx,sqrt(u(i,nz)*u(i,nz)*su*su
     &                        +w(i,nz)*w(i,nz)*sw*sw)
        WRITE(43,*) xx,surfx(i)*sigb
        WRITE(44,*) xx,surfz(i)*sigb
        WRITE(45,*) xx,bedx(i)*sigb
        WRITE(46,*) xx,bedz(i)*sigb
      ENDDO
      DO i=41,46
        WRITE(i,*) -99999,0
        WRITE(i,*) 't=',dt*(k-1)*st
      ENDDO
      IF(k.eq.nt) THEN
        DO i=41,46
          CLOSE(i)
        ENDDO
      ENDIF
      RETURN
      END
C-----------------------------------------------------------------------
      SUBROUTINE scales(delta,sx,sz,sp,su,sw,smu,st,ssig,ssigb)
C-----------------------------------------------------------------------
C     Defines scales.
C-----------------------------------------------------------------------
      IMPLICIT REAL*8(a-h,o-z)
      include "parameter2d.h"
C ---
      n    = 3.                  ! Glen's parameter
      rho  = 910.                ! [kg/m^3]
      g    = 9.81                ! [m/sec^2]
      A    = 1.e-16              ! [Pa^(-n)a^(-1)]
      spyr = 31556926.           ! [sec/a]
      sx    = Lx                 ! [m]
      sz    = 1000.              ! [m]
      delta = sz/sx              ! aspect ratio
C ---
      IF(dimensional) THEN
        sp    = rho*g*sz           ! [Pa=kg/m/sec^2]
        su    = (2*sp)**n*sx*A     ! [m/a]
        sw    = su*sz/sx           ! [m/a]
```

```
      smu   = 0.5*A**(-1./n)*(su/sx)**((1.-n)/n)  ! [Pa sec^2]
      st    = sx/su              ! [a]
      ssig  = sp/sz              ! stress scale [J=Pa/m]
      ssigb = sp                 ! bound.stress scale [Pa]
C ---
      su    = su*0.001            !  in km/a
      sw    = sw*0.001            !  in km/a
      sp    = sp*0.001            !  in kPa
      ssig  = ssig*0.001          !  in kJ
      ssibg = ssigb*0.001         !  in kPa
    ELSE
      sx=1.
      sz=1.
      sp=1.
      su=1.
      sw=1.
      smu=1.
      st=1.
      ssig=1.
      ssigb=1.
    ENDIF
    RETURN
    END
C---------------------------------------------------------------------
```

*Acknowledgements.* This work was supported by the CReSIS. We thank William Bray and Jane Morse for helping with proofreading this paper.

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**Table 1.** Constants for the benchmark experiments.

| | Constant | Value | Units |
|---|---|---|---|
| $A$ | Ice-flow parameter | $10^{-16}$ | $\mathrm{Pa^{-n}\,a^{-1}}$ |
| $\rho$ | Ice density | 910 | $\mathrm{kg\,m^{-1}}$ |
| $g$ | Gravitational constant | 9.81 | $\mathrm{m\,s^{-2}}$ |
| $n$ | Exponent in Glen's flow law | 3 | |
| | Seconds per year | 31 556 926 | $\mathrm{s\,a^{-1}}$ |



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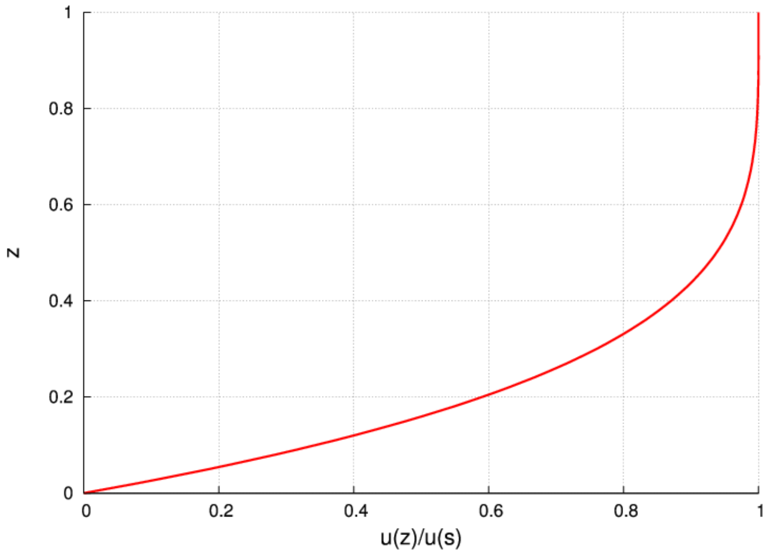


**Fig. 1.** 2-D flowline steady-state manufactured solution (coefficient $\lambda = 4$): horizontal component of velocity scaled to the surface velocity as a function of dimensionless thickness. Horizontal velocity increases with the fourth power of ice thickness. Most shearing ice concentrated near the glacier base which is similar to lamellar flow.



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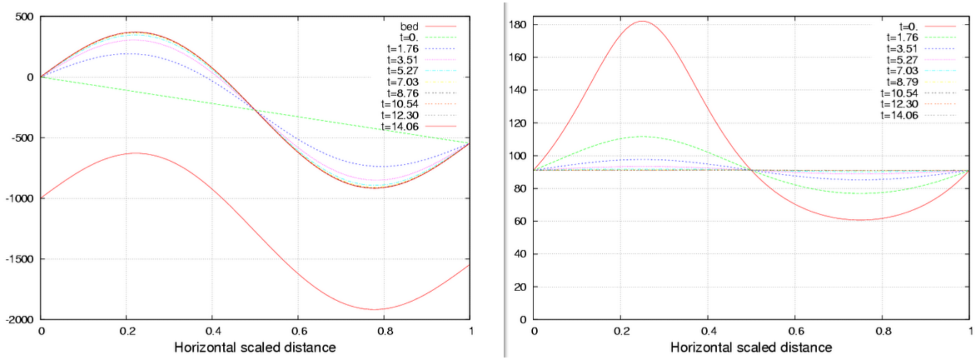


**Fig. 2.** 2-D flowline time dependent experiment - ice stream flow over bumpy bed. The left graph shows the steady bed and transformation over time of the ice surface. The right graph shows transformation over time of the norm of the surface velocity (in km/a).

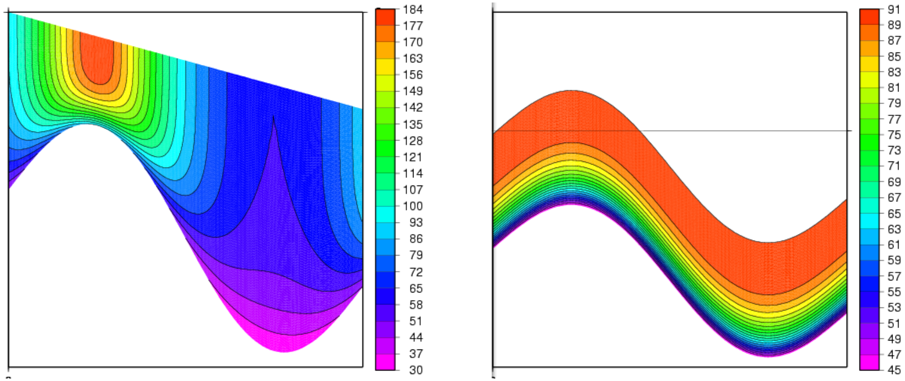


**Fig. 3.** 2-D flowline time dependent experiment – ice stream flow over bumpy bed. The graphs show the horizontal velocity $u$ at the beginning (left) and at the time when the steady-state solution is reached (right). At the beginning, the horizontal velocity is anti-correlated with ice thickness: it is larger over the bump than over the trough. At the steady-state, the horizontal velocity is spatially uniform and increases from the bed to the surface with power of $\lambda = 4$ of the ice thickness.



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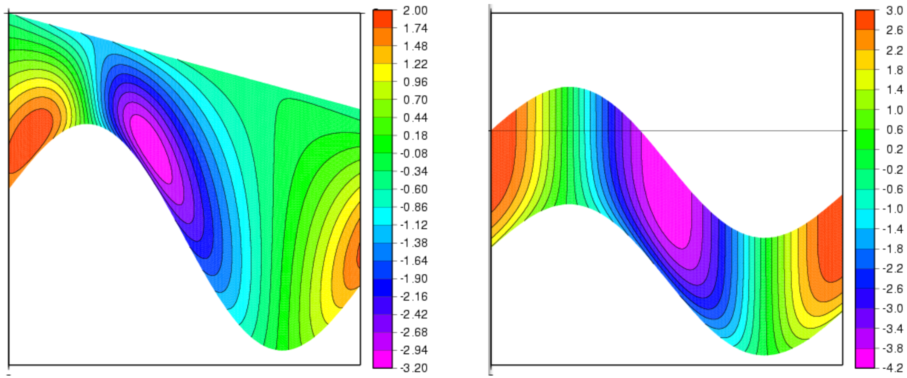


**Fig. 4.** 2-D flowline time dependent experiment - ice stream flow over bumpy bed. The graphs show the vertical velocity $w$ at the beginning (left) and at the time when the steady-state solution is reached (right). At the beginning, the vertical velocity is largest at the bed where ice shearing is the largest. At the steady-state, the vertical velocity is almost uniform in every vertical slide. This is consistent with ice-stream flow.



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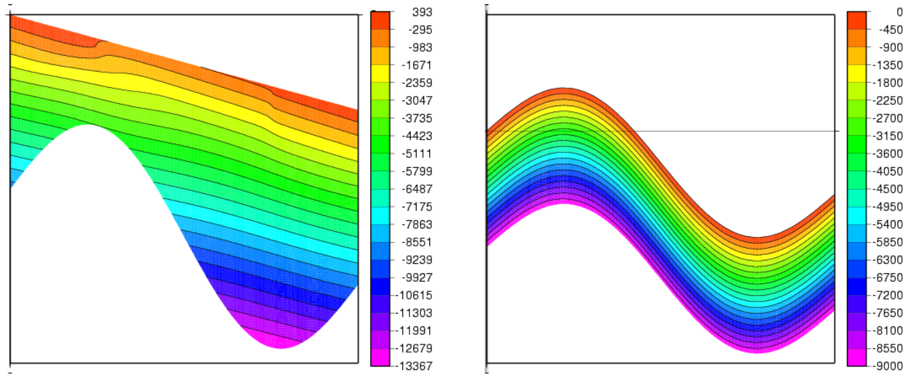


**Fig. 5.** 2-D flowline time dependent experiment - ice stream flow over bumpy bed. The graphs show ice pressure $p$ at the beginning (left) and at the time when the steady-state solution is reached (right). The ice pressure is proportional to the ice thickness. At the steady-state, it is equal zero at the ice surface.



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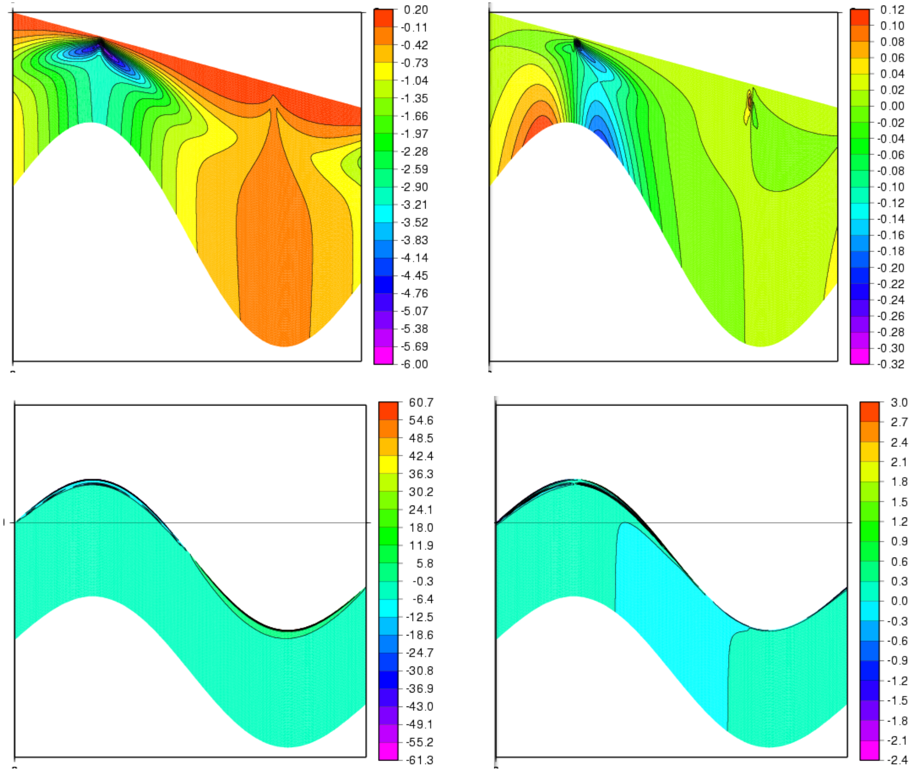


**Fig. 6.** 2-D flowline time dependent experiment – ice stream flow over bumpy bed. The graphs show the compensatory horizontal (left) and vertical (right) stress terms in the conservation of momentum equation at the beginning (top) and at the time when the steady-state solution is reached (bottom). Stress terms are in kJ. The graphs show that at the beginning both stress terms have largest values over the bump. At the steady-state solution, the stress terms are zeroes almost everywhere except a small surface layer.

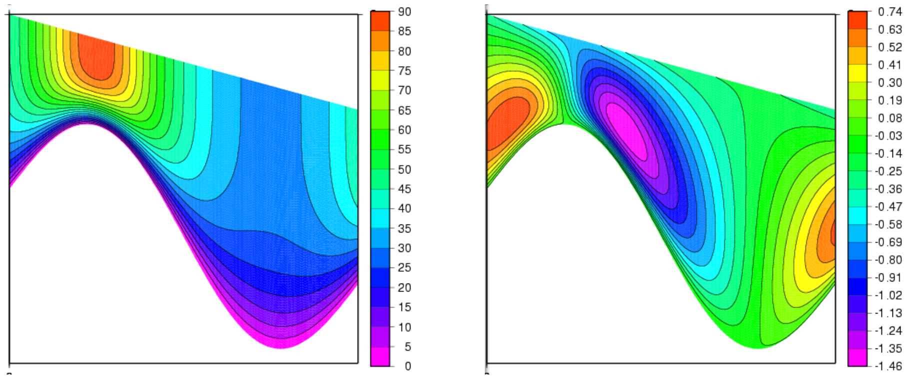


**Fig. 7.** 2-D flowline steady-state experiment – version of experiment B from (Pattyn et al., 2008) (flow with a linear sloping surface and a sinusoidal *frozen* bed). The graph shows the horizontal (left) and vertical (right) velocity fields. The horizontal velocity is anti-correlated with ice thickness.



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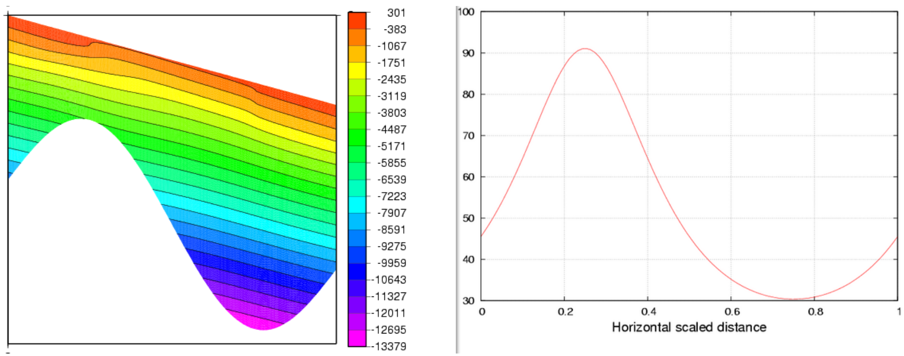


**Fig. 8.** 2-D flowline steady-state experiment – version of experiment B from (Pattyn et al., 2008) (flow with a linear sloping surface and a sinusoidal *frozen* bed). The graph shows ice pressure (left) and the norm of the surface velocity (right). The surface velocities are larger over the bump and smaller over the trough. This is consistent with the observation that in 2-D flowline experiments the ice cannot flow around the bumps. Pressure is proportional to the ice thickness.



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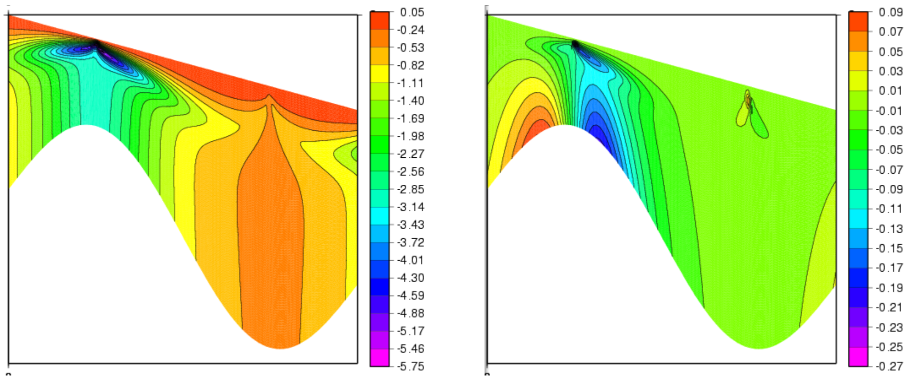


**Fig. 9.** 2-D flowline steady-state experiment – version of experiment B from (Pattyn et al., 2008) (flow with a linear sloping surface and a sinusoidal *frozen* bed). The graph shows the horizontal (left) and vertical (right) compensatory stresses $\Sigma_x$ and $\Sigma_z$ in the conservation of momentum equations (in kJ).



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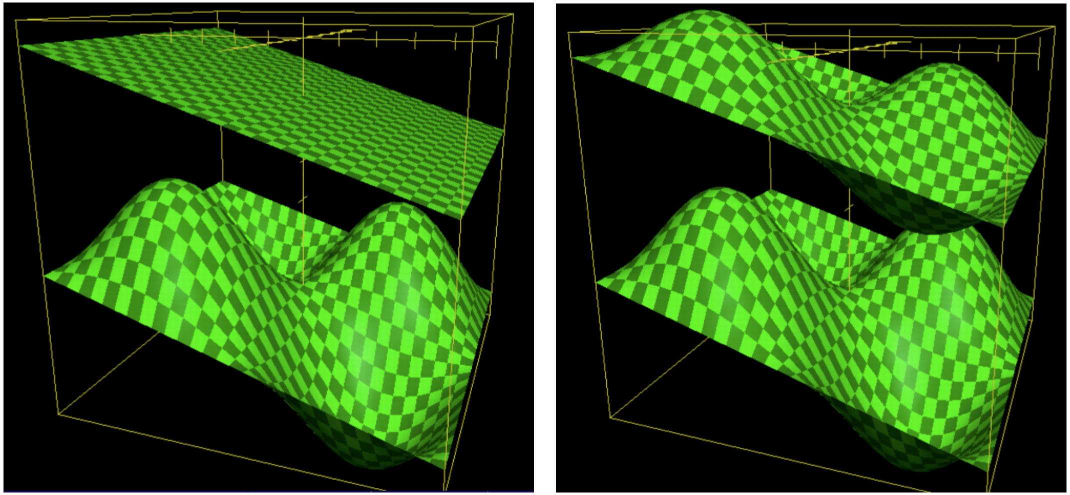

**Fig. 10.** 3-D time dependent experiment – ice flow over a bumpy bed. The top graph shows the bed and ice surface at the beginning. All distances are scaled. The bottom graph shows the bed and ice surface at the steady state. Ice flow is from left to right. The ice surface changes from a linear sloping surface to the surface draped over the topography of the bed. Ice thickness is spatially uniform when the steady-state solution is reached.



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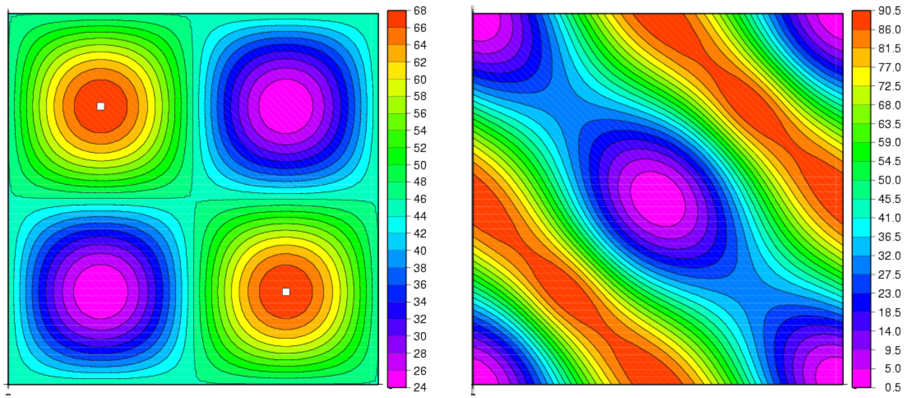


**Fig. 11.** 3-D time-dependent experiment. The left and right graphs show the ice surface $x$- and $y$-horizontal velocity respectively at the beginning. At the time when the steady-state solution is reached, both velocities at the surface are uniform and have values of 46 km/a.

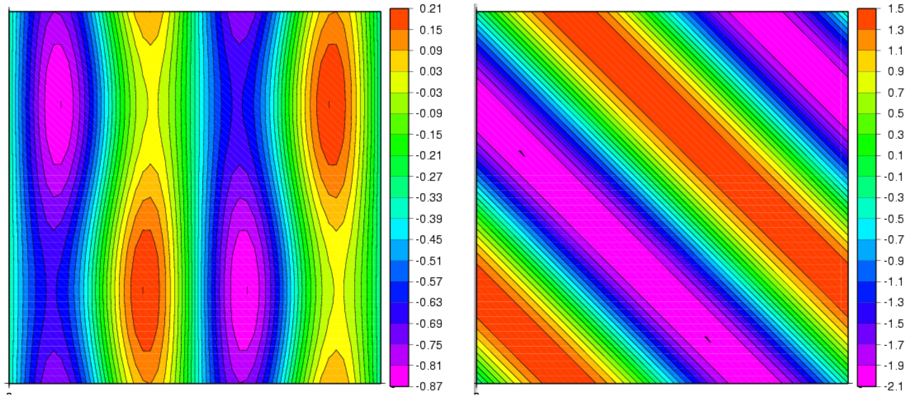


**Fig. 12.** 3-D time-dependent experiment. The graphs show the ice surface vertical velocity at the beginning (left) and at the time when the steady-state solution is reached (right).

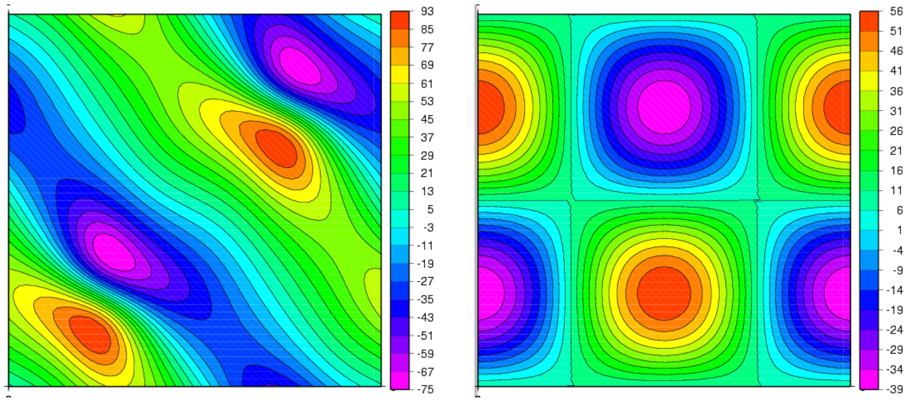


**Fig. 13.** 3-D time-dependent experiment. The graphs show the ice surface pressure at the beginning (left) and at the time when the steady-state solution is reached (right).



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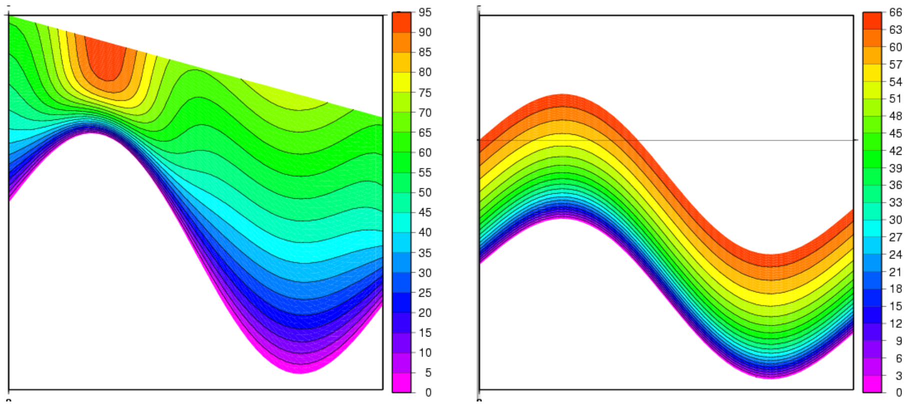


**Fig. 14.** 3-D time-dependent experiment. The graphs show the the norm of the velocity along the $y = 1/4$ slide at the beginning (left) and at the time when the steady-state is reached (right). At the beginning, velocity has two local maximums, over the bump and over the bed where the bed changes the most. At the steady-state position, velocity is spatially uniform and proportional to the ice thickness.



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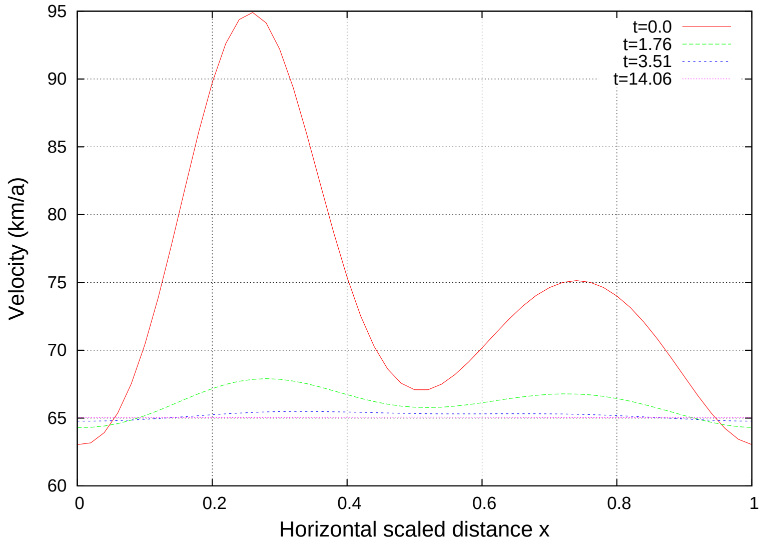


**Fig. 15.** 3-D time-dependent experiment. The graph shows the transformation over time of the norm of the surface velocity along $y = 1/4$ slide. At the beginning, velocity has two local maximums, over the bump and over the bed where the bed changes the most. At the steady-state position, the norm of the surface velocity is spatially uniform.