Supplement of The Cryosphere, 18, 4285–4313, 2024
https://doi.org/10.5194/tc-18-4285-2024-supplement

The Cryosphere
Open Access

Supplement of

Multiscale modeling of heat and mass transfer in dry snow: influence of the condensation coefficient and comparison with experiments

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S.1 Case A

Taking the order of magnitude of the dimensionless numbers into account, $\left[\mathrm{F}_i^T\right] = \mathcal{O}\left(\left[\mathrm{F}_a^T\right]\right) = \mathcal{O}\left([\mathrm{F}_a^\rho]\right) = \mathcal{O}(\varepsilon^2)$, $[\mathrm{K}] = \mathcal{O}\left(1\right), [\mathrm{H}] = \mathcal{O}\left(\varepsilon^2\right), [\mathrm{W_R}] = \mathcal{O}\left(\varepsilon^2\right)$, the dimensionless microscopic description (13)-(18) becomes:

$$\varepsilon^2 \rho_i^* C_i^* \frac{\partial T_i^*}{\partial t^*} - \mathrm{div}^*(k_i^* \mathbf{grad}^* T_i^*) = 0 \quad \text{in } \Omega_i \tag{S.A.1}$$

$$\varepsilon^2 \rho_a^* C_a^* \frac{\partial T_a^*}{\partial t^*} - \mathrm{div}^*(k_a^* \mathbf{grad} T_a^*) = 0 \quad \text{in } \Omega_a \tag{S.A.2}$$

$$\varepsilon^2 \frac{\partial \rho_v^*}{\partial t^*} - \mathrm{div}^*(D_v^* \mathbf{grad}^* \rho_v^*) = 0 \quad \text{in } \Omega_a \tag{S.A.3}$$

$$T_i^* = T_a^* \quad \text{on } \Gamma \tag{S.A.4}$$

$$k_i^* \mathbf{grad}^* T_i^* \cdot \mathbf{n_i} - k_a^* \mathbf{grad}^* T_a^* \cdot \mathbf{n_i} = \varepsilon^2 L_{sg}^* \mathbf{w}^* \cdot \mathbf{n_i} \quad \text{on } \Gamma \tag{S.A.5}$$

$$D_v^* \mathbf{grad}^* \rho_v^* \cdot \mathbf{n_i} = \varepsilon^2 \rho_i^* \mathbf{w}^* \cdot \mathbf{n_i} \quad \text{on } \Gamma \tag{S.A.6}$$

This set of equations is completed by the Hertz-Knudsen equation (10) and the Clausius Clapeyron's law (9) expressed in dimensionless form as:

$$w_n^* = \mathbf{w}^* \cdot \mathbf{n_i} = \frac{\alpha^*}{\rho_i^*} w_\mathrm{k}^* \left[\rho_v^* - \rho_{vs}^*(T_a^*)\right] \;\; \text{on } \Gamma \tag{S.A.7}$$

$$\rho_{vs}^*(T_a^*) = \rho_{vs}^{\mathrm{ref}*}(T^{\mathrm{ref}*}) \exp\left[\frac{L_{sg}^* m^*}{\rho_i^* k^*}\left(\frac{1}{T^{\mathrm{ref}*}} - \frac{1}{T_a^*}\right)\right] \tag{S.A.8}$$

S.1.1 Heat transfer

Introducing asymptotic expansions for T_i^* and T_a^* in the relations (S.A.1), (S.A.2), (S.A.4), (S.A.5) gives at the lowest order:

$$\mathrm{div}_{y^*}(k_i^* \mathbf{grad}_{y^*} T_i^{*(0)}) = 0 \quad \text{in } \Omega_i \tag{S.A.9}$$

$$\mathrm{div}_{y^*}(k_a^* \mathbf{grad}_{y^*} T_a^{*(0)}) = 0 \quad \text{in } \Omega_a \tag{S.A.10}$$

$$T_i^{*(0)} = T_a^{*(0)} \quad \text{on } \Gamma \tag{S.A.11}$$

$$(k_i^* \mathbf{grad}_{y^*}^* T_i^{*(0)} - k_a^* \mathbf{grad}_{y^*}^* T_a^{*(0)}) \cdot \mathbf{n_i} = 0 \quad \text{on } \Gamma \tag{S.A.12}$$

where the unknowns $T_i^{*(0)}(\mathbf{x}^*,\mathbf{y}^*,t)$ and $T_a^{*(0)}(\mathbf{x}^*,\mathbf{y}^*,t)$ are $\mathbf{y}^*$-periodic. It can be shown that the obvious solution of the above boundary value problem is given by:

$$T_i^{*(0)} = T_a^{*(0)} = T^{*(0)}(\mathbf{x}^*,t) \tag{S.A.13}$$

At the first order, the temperature is independent of the microscopic dimensionless variable $\mathbf{y}^*$, i.e. we have only one temperature field. Taking these results into account, equations (S.A.1), (S.A.2), (S.A.4), and (S.A.5) of order ε give the following second-order problem:

$$\mathrm{div}_{y^*}(k_i^*(\mathbf{grad}_{y^*}T_i^{*(1)} + \mathbf{grad}_{x^*}T^{*(0)})) = 0 \quad \text{in } \Omega_i \tag{S.A.14}$$

$$\mathrm{div}_{y^*}(k_a^*(\mathbf{grad}_{y^*}T_a^{*(1)} + \mathbf{grad}_{x^*}T^{*(0)})) = 0 \quad \text{in } \Omega_a \tag{S.A.15}$$

$$T_i^{*(1)} = T_a^{*(1)} \quad \text{on } \Gamma \tag{S.A.16}$$

$$(k_i^*(\mathbf{grad}_{y^*}T_i^{*(1)} + \mathbf{grad}_{x^*}T^{*(0)}) - k_a^*(\mathbf{grad}_{y^*}T_a^{*(1)} + \mathbf{grad}_{x^*}T^{*(0)}))\cdot\mathbf{n_i} = 0 \quad \text{on } \Gamma \tag{S.A.17}$$

where the unknowns $T_i^{*(1)}(\mathbf{x}^*,\mathbf{y}^*,t)$ and $T_a^{*(1)}(\mathbf{x}^*,\mathbf{y}^*,t)$ are $\mathbf{y}^*$-periodic and the macroscopic gradient $\mathbf{grad}_{x^*}T^{*(0)}$ is given. The solution of the above boundary value problem appears as a linear function of the macroscopic gradient, modulo an arbitrary function $\widetilde{T}^{*(1)}(\mathbf{x}^*,t)$:

$$T_i^{*(1)}(\mathbf{x}^*,\mathbf{y}^*,t) = \mathbf{t}_i^*(\mathbf{y}^*)\cdot\mathbf{grad}_{x^*}T^{*(0)} + \widetilde{T}_i^{*(1)} \tag{S.A.18}$$

$$T_a^{*(1)}(\mathbf{x}^*,\mathbf{y}^*,t) = \mathbf{t}_a^*(\mathbf{y}^*)\cdot\mathbf{grad}_{x^*}T^{*(0)} + \widetilde{T}_a^{*(1)} \tag{S.A.19}$$

where $\mathbf{t}_i^*(\mathbf{y}^*)$ and $\mathbf{t}_a^*(\mathbf{y}^*)$ are two periodic vectors which characterize the fluctuation of temperature in both phases at the pore scale. Introducing (S.A.18) and (S.A.19) in the set (S.A.14)-(S.A.17), these two vectors are solution of the following boundary value problem, expressed in a compact form as:

$$\mathrm{div}_{y^*}(k_i^*(\mathbf{grad}_{y^*}\mathbf{t}_i^* + \mathbf{I})) = 0 \quad \text{in } \Omega_i \tag{S.A.20}$$

$$\mathrm{div}_{y^*}(k_a^*(\mathbf{grad}_{y^*}\mathbf{t}_a^* + \mathbf{I})) = 0 \quad \text{in } \Omega_a \tag{S.A.21}$$

$$\mathbf{t}_i^* = \mathbf{t}_a^* \quad \text{on } \Gamma \tag{S.A.22}$$

$$(k_i^*(\mathbf{grad}_{y^*}\mathbf{t}_i^* + \mathbf{I}) - k_a^*(\mathbf{grad}_{y^*}\mathbf{t}_a^* + \mathbf{I}))\cdot\mathbf{n_i} = 0 \quad \text{on } \Gamma \tag{S.A.23}$$

$$\frac{1}{|\Omega|}\int_{\Omega}(\mathbf{t}_\mathrm{a}^* + \mathbf{t}_\mathrm{i}^*)\mathrm{d}\Omega = \mathbf{0} \tag{S.A.24}$$

This latter equation is introduced to ensure the uniqueness of the solution. Finally, the third order problem is given by the Eq. (S.A.1), (S.A.2), (S.A.4), and (S.A.5) of order ε^2:

$$\rho_i^* C_i^* \frac{\partial T^{*(0)}}{\partial t^*} - \mathrm{div}_{y^*}(k_i^*(\mathbf{grad}_{y^*} T_i^{*(2)} + \mathbf{grad}_{x^*} T_i^{*(1)})) - \mathrm{div}_{x^*}(k_i^*(\mathbf{grad}_{y^*} T_i^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)})) = 0 \quad \text{in } \Omega_i \qquad \text{(S.A.25)}$$

$$\rho_a^* C_a^* \frac{\partial T^{*(0)}}{\partial t^*} - \mathrm{div}_{y^*}(k_a^*(\mathbf{grad}_{y^*} T_a^{*(2)} + \mathbf{grad}_{x^*} T_a^{*(1)})) - \mathrm{div}_{x^*}(k_a^*(\mathbf{grad}_{y^*} T_a^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)})) = 0 \quad \text{in } \Omega_a \qquad \text{(S.A.26)}$$

$$T_i^{*(2)} = T_a^{*(2)} \quad \text{on } \Gamma \qquad \text{(S.A.27)}$$

$$(k_i^*(\mathbf{grad}_{y^*} T_i^{*(2)} + \mathbf{grad}_{x^*} T_i^{*(1)}) - k_a^*(\mathbf{grad}_{y^*} T_a^{*(2)} + \mathbf{grad}_{x^*} T_a^{*(1)})) \cdot \mathbf{n_i} = w_n^{*(0)} \quad \text{on } \Gamma \qquad \text{(S.A.28)}$$

where the unknowns $T_i^{*(2)}(\mathbf{x}^*, \mathbf{y}^*, t)$ and $T_a^{*(2)}(\mathbf{x}^*, \mathbf{y}^*, t)$ are $\mathbf{y}^*$-periodic and $w_n^{*(0)}$ is the normal interface velocity due to the sublimation-deposition process given, at the zero order, by the Hertz-Knudsen equation (S.A.7) and the Clausius Clapeyron's law (S.A.8).

S.1.2 Water vapor transfer

Introducing asymptotic expansions for ρ_v^* in the relations (S.A.3) and (S.A.6) gives at the lowest order:

$$\mathrm{div}_{y^*}(D_v^* \mathbf{grad}_{y^*} \rho_v^{*(0)}) = 0 \quad \text{in } \Omega_a \qquad \text{(S.A.29)}$$

$$D_v^* \mathbf{grad}_{y^*} \rho_v^{*(0)} \cdot \mathbf{n_i} = 0 \quad \text{on } \Gamma \qquad \text{(S.A.30)}$$

where the unknown $\rho_v^{*(0)}(\mathbf{x}^*, \mathbf{y}^*, t)$ is $\mathbf{y}^*$-periodic. It can be shown (Auriault et al., 2009) that the solution of the above boundary value problem is given by:

$$\rho_v^{*(0)} = \rho_v^{*(0)}(\mathbf{x}^*, t) \qquad \text{(S.A.31)}$$

At the first order, the water vapor density is independent of the microscopic dimensionless variable $\mathbf{y}^*$. Taking these results into account, the second-order problem is given by Eq. (S.A.3) and (S.A.6) of order ε, which are:

$$\mathrm{div}_{y^*}(D_v^*(\mathbf{grad}_{y^*} \rho_v^{*(1)} + \mathbf{grad}_{x^*} \rho_v^{*(0)})) = 0 \quad \text{in } \Omega_a \qquad \text{(S.A.32)}$$

$$D_v^*(\mathbf{grad}_{y^*} \rho_v^{*(1)} + \mathbf{grad}_{x^*} \rho_v^{*(0)}) \cdot \mathbf{n_i} = 0 \quad \text{on } \Gamma \qquad \text{(S.A.33)}$$

where the unknown $\rho_v^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t)$ is $\mathbf{y}^*$-periodic and the macroscopic gradient $\mathbf{grad}_{x^*} \rho_v^{*(0)}$ is given. The solution of the above boundary value problem appears as a linear function of the macroscopic gradient, modulo an arbitrary function $\widetilde{\rho}_v^{*(1)}(\mathbf{x}^*, t)$ (Auriault et al., 2009):

$$\rho_v^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t) = \mathbf{g}_v^*(\mathbf{y}^*) \cdot \mathbf{grad}_{x^*} \rho_v^{*(0)} + \widetilde{\rho}_v^{*(1)}(\mathbf{x}^*, t) \qquad \text{(S.A.34)}$$

where $\mathbf{g}_v^*(\mathbf{y}^*)$ is a periodic vector which characterizes the fluctuation of water vapor density in the air phase at the pore scale. Introducing (S.A.34) in the set (S.A.32)-(S.A.33), this vector is solution of the following boundary value problem, expressed in a compact form:

$$\text{div}_{y^*}\left(D_v^*(\mathbf{grad}_{y^*}\mathbf{g}_v^* + \mathbf{I})\right) = 0 \quad \text{in } \Omega_a \tag{S.A.35}$$

$$D_v^*(\mathbf{grad}_{y^*}\mathbf{g}_v^* + \mathbf{I}) \cdot \mathbf{n_i} = 0 \quad \text{on } \Gamma \tag{S.A.36}$$

$$\frac{1}{|\Omega|}\int_{\Omega_a} \mathbf{g}_v^* \text{d}\Omega = \mathbf{0} \tag{S.A.37}$$

This latter equation is introduced to ensure the uniqueness of the solution. Finally, the third order problem is given by Eq. (S.A.3) and (S.A.6) of order ε^2:

$$\frac{\partial \rho_v^{*(0)}}{\partial t^*} - \text{div}_{y^*}\left(D_v^*(\mathbf{grad}_{y^*}\rho_v^{*(2)} + \mathbf{grad}_{x^*}\rho_v^{*(1)})\right) - \text{div}_{x^*}\left(D_v^*(\mathbf{grad}_{y^*}\rho_v^{*(1)} + \mathbf{grad}_{x^*}\rho_v^{*(0)})\right) = 0 \quad \text{in } \Omega_a \tag{S.A.38}$$

$$D_v^*(\mathbf{grad}_{y^*}\rho_v^{*(2)} + \mathbf{grad}_{x^*}\rho_v^{*(1)}) \cdot \mathbf{n_i} = w_n^{*(0)} \quad \text{on } \Gamma \tag{S.A.39}$$

where the unknown $\rho_v^{*(2)}(\mathbf{x}^*, \mathbf{y}^*, t)$ is $\mathbf{y}^*$-periodic and $w_n^{*(0)}$ is the normal interface velocity due to the sublimation/deposition process given, at the zero order, by the Hertz-Knudsen equation (S.A.7) and the Clausius Clapeyron's law (S.A.8). Taking the above results into account, we have:

$$\rho_{vs}^*(T_a^*) = \rho_{vs}^{\text{ref}*}(T^{\text{ref}*}) \exp\left[\frac{L_{sg}^* m^*}{\rho_i^* k^*}\left(\frac{1}{T^{\text{ref}*}} - \frac{1}{T^{*(0)}}\right)\right]\left(1 + \varepsilon \frac{L_{sg}^* m^*}{\rho_i^* k^*}\frac{T_a^{*(1)}}{(T^{*(0)})^2} + \ldots\right) \tag{S.A.40}$$

This relation shows that the asymptotic development of the Clausius-Clapeyron's law is written:

$$\rho_{vs}^*(T_a^*) = \rho_{vs}^{*(0)}(\mathbf{x}^*, t) + \varepsilon \rho_{vs}^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t) + \ldots \tag{S.A.41}$$

where the first term $\rho_{vs}^{*(0)}$, which depends on $T^{*(0)}(\mathbf{x}^*, t)$ only, is defined as:

$$\rho_{vs}^{*(0)}(T^{*(0)}) = \rho_{vs}^{\text{ref}*}(T^{\text{ref}*}) \exp\left[\frac{L_{sg}^* m^*}{\rho_i^* k^*}\left(\frac{1}{T^{\text{ref}*}} - \frac{1}{T^{*(0)}}\right)\right] \tag{S.A.42}$$

The relation (S.A.42) shows that the normal velocity $w_n^{*(0)}$ arising in the interface condition (S.A.39) does not depend on $\mathbf{y}^*$. From (S.A.7), $w_n^{*(0)}$ is also written:

$$w_n^{*(0)} = \frac{\alpha^*}{\rho_i^*} w_\text{k}^* \left[\rho_v^{*(0)} - \rho_{vs}^{*(0)}(T^{*(0)})\right] \tag{S.A.43}$$

S.1.3 Macroscopic description

Integrating (S.A.25) over Ω_i and (S.A.26) over Ω_a, and then using the divergence theorem, the periodicity condition, and the interface conditions (S.A.28) leads to the first order dimensionless description:

$$(\rho C)^{\text{eff}*} \frac{\partial T^{*(0)}}{\partial t^*} - \text{div}_{x^*}(\mathbf{k}^{\text{eff}*}\mathbf{grad}_{x^*}\, T^{*(0)}) = \text{SSA}_\text{V} L_{sg}^* w_n^{*(0)} \tag{S.A.44}$$

where $\mathrm{SSA_V} = |\Gamma|/|\Omega|$ is the specific surface area and where $(\rho C)^{\mathrm{eff}*}$ and $\mathbf{k}^{\mathrm{eff}*}$ are the dimensionless effective thermal capacity and the effective dimensionless thermal conductivity, respectively, defined as:

$$(\rho C)^{\mathrm{eff}*} = (1-\phi)\rho_i^* C_i^* + \phi \rho_a^* C_a^* \tag{S.A.45}$$

$$\mathbf{k}^{\mathrm{eff}*} = \frac{1}{|\Omega|}\left(\int\limits_{\Omega_\mathrm{a}} k_a^*(\mathbf{grad}_{y^*}\mathbf{t}_a^*(\mathbf{y}^*) + \mathbf{I})\mathrm{d}\Omega + \int\limits_{\Omega_\mathrm{i}} k_i^*(\mathbf{grad}_{y^*}\mathbf{t}_i^*(\mathbf{y}^*) + \mathbf{I})\mathrm{d}\Omega\right) \tag{S.A.46}$$

where ϕ is the porosity. Consequently, integrating (S.A.38) over Ω_a, and then using the divergence theorem, the periodicity condition, and the interface conditions (S.A.39) leads to the first order dimensionless description:

$$\phi \frac{\partial \rho_v^{*(0)}}{\partial t} - \mathrm{div}_{x^*}(\mathbf{D}^{\mathrm{eff}*}\mathbf{grad}_{x^*}\rho_v^{*(0)}) = -\mathrm{SSA_V}\rho_i^* w_n^{*(0)} \tag{S.A.47}$$

where $\mathbf{D}^{\mathrm{eff}*}$ is the dimensionless effective diffusion tensor defined as:

$$\mathbf{D}^{\mathrm{eff}*} = \frac{1}{|\Omega|}\int\limits_{\Omega_a} D_v^*(\mathbf{grad}_{y^*}\ \mathbf{g}_v^*(\mathbf{y}^*) + \mathbf{I})\mathrm{d}\,\Omega \tag{S.A.48}$$

S.2 Case B

Taking into account the order of magnitude of the dimensionless numbers, $[\mathrm{F}_i^T] = \mathcal{O}\left([\mathrm{F}_a^T]\right) = \mathcal{O}([\mathrm{F}_a^\rho]) = \mathcal{O}(\varepsilon^2)$, $[\mathrm{K}] = \mathcal{O}(1)$, $[\mathrm{H}] = \mathcal{O}(\varepsilon)$, $[\mathrm{W_R}] = \mathcal{O}(\varepsilon)$, the dimensionless microscopic description (13)-(18) becomes:

$$\varepsilon^2 \rho_i^* C_i^* \frac{\partial T_i^*}{\partial t^*} - \mathrm{div}^*(k_i^*\mathbf{grad}^* T_i^*) = 0 \quad \text{in } \Omega_i \tag{S.B.1}$$

$$\varepsilon^2 \rho_a^* C_a^* \frac{\partial T_a^*}{\partial t^*} - \mathrm{div}^*(k_a^*\mathbf{grad} T_a^*) = 0 \quad \text{in } \Omega_a \tag{S.B.2}$$

$$\varepsilon^2 \frac{\partial \rho_v^*}{\partial t^*} - \mathrm{div}^*(D_v^*\mathbf{grad}^*\rho_v^*) = 0 \quad \text{in } \Omega_a \tag{S.B.3}$$

$$T_i^* = T_a^* \quad \text{on } \Gamma \tag{S.B.4}$$

$$k_i^*\mathbf{grad}^* T_i^* \cdot \mathbf{n_i} - k_a^*\mathbf{grad}^* T_a^* \cdot \mathbf{n_i} = \varepsilon L_{sg}^* \mathbf{w}^* \cdot \mathbf{n_i} \quad \text{on } \Gamma \tag{S.B.5}$$

$$D_v^*\mathbf{grad}^*\rho_v^* \cdot \mathbf{n_i} = \varepsilon \rho_i^* \mathbf{w}^* \cdot \mathbf{n_i} \quad \text{on } \Gamma \tag{S.B.6}$$

This set of equations is completed by the Hertz-Knudsen equation (10) and the Clausius Clapeyron's law (9) expressed in dimensionless form as:

$$w_n^* = \mathbf{w}^* \cdot \mathbf{n_i} = \frac{\alpha^*}{\rho_i^*} w_\mathrm{k}^* \left[\rho_v^* - \rho_{vs}^*(T_a^*)\right] \quad \text{on } \Gamma \tag{S.B.7}$$

$$\rho_{vs}^*(T_a^*) = \rho_{vs}^{\mathrm{ref}*}(T^{\mathrm{ref}*}) \exp\left[\frac{L_{sg}^* m^*}{\rho_i^* k^*}\left(\frac{1}{T^{\mathrm{ref}*}} - \frac{1}{T_a^*}\right)\right] \tag{S.B.8}$$

S.2.1 Heat transfer

Introducing asymptotic expansions for T_i^* and T_a^* in the relations (S.B.1), (S.B.2), (S.B.4), and (S.B.5) gives at the lowest order:

$$\mathrm{div}_{y^*}(k_i^* \mathbf{grad}_{y^*} T_i^{*(0)}) = 0 \quad \text{in } \Omega_i \tag{S.B.9}$$

$$\mathrm{div}_{y^*}(k_a^* \mathbf{grad}_{y^*} T_a^{*(0)}) = 0 \quad \text{in } \Omega_a \tag{S.B.10}$$

$$T_i^{*(0)} = T_a^{*(0)} \quad \text{on } \Gamma \tag{S.B.11}$$

$$(k_i^* \mathbf{grad}_{y^*}^* T_i^{*(0)} - k_a^* \mathbf{grad}_{y^*}^* T_a^{*(0)}) \cdot \mathbf{n_i} = 0 \quad \text{on } \Gamma \tag{S.B.12}$$

where the unknowns $T_i^{*(0)}(\mathbf{x}^*, \mathbf{y}^*, t)$ and $T_a^{*(0)}(\mathbf{x}^*, \mathbf{y}^*, t)$ are $\mathbf{y}^*$-periodic. It can be shown (Auriault et al., 2009) that the obvious solution of the above boundary value problem is given by:

$$T_i^{*(0)} = T_a^{*(0)} = T^{*(0)}(\mathbf{x}^*, t) \tag{S.B.13}$$

At the first order, the temperature is independent of the microscopic dimensionless variable $\mathbf{y}^*$, i.e. we have only one temperature field. Taking these results into account, Eq. (S.B.1), (S.B.2), (S.B.4), and (S.B.5) of order ε give the following second-order problem:

$$\mathrm{div}_{y^*}(k_i^* (\mathbf{grad}_{y^*} T_i^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)})) = 0 \quad \text{in } \Omega_i \tag{S.B.14}$$

$$\mathrm{div}_{y^*}(k_a^* (\mathbf{grad}_{y^*} T_a^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)})) = 0 \quad \text{in } \Omega_a \tag{S.B.15}$$

$$T_i^{*(1)} = T_a^{*(1)} \quad \text{on } \Gamma \tag{S.B.16}$$

$$(k_i^* (\mathbf{grad}_{y^*} T_i^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)}) - k_a^* (\mathbf{grad}_{y^*} T_a^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)})) \cdot \mathbf{n_i} = L_{sg}^* w_n^{*(0)} \quad \text{on } \Gamma \tag{S.B.17}$$

where the unknowns $T_i^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t)$ and $T_a^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t)$ are $\mathbf{y}^*$-periodic and the macroscopic gradient $\mathbf{grad}_{x^*} T^{*(0)}$ is given. Moreover, it can be shown that at the first order $w_n^{*(0)} = 0$ (see S.B.37). As in the case A, the solution of the above boundary value problem appears as a linear function of the macroscopic gradient, modulo an arbitrary function $\widetilde{T}^{*(1)}(\mathbf{x}^*, t)$ (Auriault et al., 2009):

$$T_i^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t) = \mathbf{t}_i^*(\mathbf{y}^*) \cdot \mathbf{grad}_{x^*} T^{*(0)} + \widetilde{T}_i^{*(1)} \tag{S.B.18}$$

$$T_a^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t) = \mathbf{t}_a^*(\mathbf{y}^*) \cdot \mathbf{grad}_{x^*} T^{*(0)} + \widetilde{T}_a^{*(1)} \tag{S.B.19}$$

where $\mathbf{t}_i^*(\mathbf{y}^*)$ and $\mathbf{t}_a^*(\mathbf{y}^*)$ are two periodic vectors which characterize the fluctuation of temperature in both phases at the pore scale. Introducing (S.B.18) and (S.B.19) in the set (S.B.14)-(S.B.17), these two vectors are solution of the following boundary value problem in a compact form:

$$\mathrm{div}_{y^*}(k_i^*(\mathbf{grad}_{y^*}\mathbf{t}_i^* + \mathbf{I})) = 0 \quad \text{in } \Omega_i \tag{S.B.20}$$

$$\mathrm{div}_{y^*}(k_a^*(\mathbf{grad}_{y^*}\mathbf{t}_a^* + \mathbf{I})) = 0 \quad \text{in } \Omega_a \tag{S.B.21}$$

$$\mathbf{t}_i^* = \mathbf{t}_a^* \quad \text{on } \Gamma \tag{S.B.22}$$

$$(k_i^*(\mathbf{grad}_{y^*}\mathbf{t}_i^* + \mathbf{I}) - k_a^*(\mathbf{grad}_{y^*}\mathbf{t}_a^* + \mathbf{I}))\cdot\mathbf{n_i} = 0 \quad \text{on } \Gamma \tag{S.B.23}$$

$$\frac{1}{|\Omega|}\int_{\Omega}(\mathbf{t}_\mathrm{a}^* + \mathbf{t}_\mathrm{i}^*)\mathrm{d}\Omega = \mathbf{0} \tag{S.B.24}$$

This latter equation is introduced to ensure the uniqueness of the solution. Finally, the third order problem is given by Eq. (S.B.1), (S.B.2), (S.B.4), and (S.B.5) of order ε^2:

$$\rho_i^* C_i^* \frac{\partial T^{*(0)}}{\partial t^*} - \mathrm{div}_{y^*}(k_i^*(\mathbf{grad}_{y^*}T_i^{*(2)} + \mathbf{grad}_{x^*}T_i^{*(1)})) - \mathrm{div}_{x^*}(k_i^*(\mathbf{grad}_{y^*}T_i^{*(1)} + \mathbf{grad}_{x^*}T^{*(0)})) = 0 \quad \text{in } \Omega_i \tag{S.B.25}$$

$$\rho_a^* C_a^* \frac{\partial T^{*(0)}}{\partial t^*} - \mathrm{div}_{y^*}(k_a^*(\mathbf{grad}_{y^*}T_a^{*(2)} + \mathbf{grad}_{x^*}T_a^{*(1)})) - \mathrm{div}_{x^*}(k_a^*(\mathbf{grad}_{y^*}T_a^{*(1)} + \mathbf{grad}_{x^*}T^{*(0)})) = 0 \quad \text{in } \Omega_a \tag{S.B.26}$$

$$T_i^{*(2)} = T_a^{*(2)} \quad \text{on } \Gamma \tag{S.B.27}$$

$$(k_i^*(\mathbf{grad}_{y^*}T_i^{*(2)} + \mathbf{grad}_{x^*}T_i^{*(1)}) - k_a^*(\mathbf{grad}_{y^*}T_a^{*(2)} + \mathbf{grad}_{x^*}T_a^{*(1)}))\cdot\mathbf{n_i} = L_{sg}^* w_n^{*(1)} \quad \text{on } \Gamma \tag{S.B.28}$$

where the unknowns $T_i^{*(2)}(\mathbf{x}^*, \mathbf{y}^*, t)$ and $T_a^{*(2)}(\mathbf{x}^*, \mathbf{y}^*, t)$ are $\mathbf{y}^*$-periodic. Integrating (S.B.25) over Ω_i and (S.B.26) over Ω_a, and then using the divergence theorem, the periodicity condition, and the boundary conditions (S.B.28) leads to the first order dimensionless description:

$$(\rho C)^{\mathrm{eff}*}\frac{\partial T^{*(0)}}{\partial t^*} - \mathrm{div}_{x^*}(\mathbf{k}^{\mathrm{eff}*}\mathbf{grad}_{x^*}T^{*(0)}) = \int_{\Gamma} L_{sg}^* w_n^{*(1)} dS = -L_{sg}^*\dot{\phi} \tag{S.B.29}$$

where $(\rho C)^{\mathrm{eff}*}$ and $\mathbf{k}^{\mathrm{eff}*}$ are the dimensionless effective thermal capacity and the effective dimensionless thermal conductivity respectively, defined, as in the Case A, by:

$$(\rho C)^{\mathrm{eff}*} = (1-\phi)\rho_i^* C_i^* + \phi\rho_a^* C_a^* \tag{S.B.30}$$

$$\mathbf{k}^{\mathrm{eff}*} = \frac{1}{|\Omega|}\left(\int_{\Omega_\mathrm{a}} k_a^*(\mathbf{grad}_{y^*}\mathbf{t}_a^*(\mathbf{y}^*) + \mathbf{I})\mathrm{d}\Omega + \int_{\Omega_\mathrm{i}} k_i^*(\mathbf{grad}_{y^*}\mathbf{t}_i^*(\mathbf{y}^*) + \mathbf{I})\mathrm{d}\Omega\right) \tag{S.B.31}$$

where ϕ is the porosity.

S.2.2 Water vapor transfer

Introducing asymptotic expansions for ρ_v^* in the relations (S.B.3) and (S.B.6) gives at the lowest order:

$$\mathrm{div}_{y^*}(D_v^* \mathbf{grad}_{y^*} \rho_v^{*(0)}) = 0 \quad \text{in } \Omega_a \tag{S.B.32}$$

$$D_v^* \mathbf{grad}_{y^*} \rho_v^{*(0)} \cdot \mathbf{n_i} = 0 \quad \text{on } \Gamma \tag{S.B.33}$$

where the unknown $\rho_v^{*(0)}(\mathbf{x}^*, \mathbf{y}^*, t)$ is $\mathbf{y}^*$-periodic. It can be shown (Auriault et al., 2009) that the solution of the above boundary value problem is given by:

$$\rho_v^{*(0)} = \rho_v^{*(0)}(\mathbf{x}^*, t) \tag{S.B.34}$$

At the first order, the water vapor density is independent of the microscopic dimensionless variable $\mathbf{y}^*$. Taking these results into account, the second-order problem is given by Eq. (S.B.3) and (S.B.6) of order ε:

$$\mathrm{div}_{y^*}(D_v^*(\mathbf{grad}_{y^*} \rho_v^{*(1)} + \mathbf{grad}_{x^*} \rho_v^{*(0)})) = 0 \quad \text{in } \Omega_a \tag{S.B.35}$$

$$D_v^*(\mathbf{grad}_{y^*} \rho_v^{*(1)} + \mathbf{grad}_{x^*} \rho_v^{*(0)}) \cdot \mathbf{n_i} = \alpha w_{\mathrm{k}} \left[\rho_v^{*(0)} - \rho_{vs}^{*(0)}(T^{*(0)})\right] \quad \text{on } \Gamma \tag{S.B.36}$$

where the unknown $\rho_v^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t)$ is $\mathbf{y}^*$-periodic. Consequently, integrating (S.B.35) over Ω_a, and then using the divergence theorem, the periodicity condition, the interface conditions (S.B.36) and the result (S.B.34) leads to the first order dimensionless description:

$$\rho_v^{*(0)} = \rho_{vs}^{*(0)}(T^{*(0)}) \tag{S.B.37}$$

Consequently, as in the Case A, the solution of the above boundary value problem (S.B.35) - (S.B.36) appears as a linear function of the macroscopic gradient $\mathbf{grad}_{x^*} \rho_{vs}^{*(0)}(T^{*(0)})$ modulo an arbitrary function $\widetilde{\rho}_v^{*(1)}(\mathbf{x}^*, t)$:

$$\rho_v^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t) = \mathbf{g}_v^*(\mathbf{y}^*) \cdot \mathbf{grad}_{x^*} \rho_{vs}^{*(0)}(T^{*(0)}) + \widetilde{\rho}_v^{*(1)}(\mathbf{x}^*, t) \tag{S.B.38}$$

where $\mathbf{g}_v^*(\mathbf{y}^*)$ is a periodic vector which characterizes the fluctuation of water vapor density in the air phase at the pore scale induced by the macroscopic gradient $\mathbf{grad}_{x^*} \rho_{vs}^{*(0)}(T^{*(0)})$. Introducing (S.B.38) in the set (S.B.35)-(S.B.36), this vector is solution of the following boundary value problem in a compact form:

$$\mathrm{div}_{y^*}(D_v^*(\mathbf{grad}_{y^*} \mathbf{g}_v^* + \mathbf{I})) = 0 \quad \text{in } \Omega_a \tag{S.B.39}$$

$$D_v^*(\mathbf{grad}_{y^*} \mathbf{g}_v^* + \mathbf{I}) \cdot \mathbf{n_i} = 0 \quad \text{on } \Gamma \tag{S.B.40}$$

$$\frac{1}{|\Omega|} \int_{\Omega_a} \mathbf{g}_v^* \mathrm{d}\Omega = \mathbf{0} \tag{S.B.41}$$

This latter equation is introduced to ensure the uniqueness of the solution. Finally, the third order problem is given by Eq. (S.B.3) and (S.B.6) of order ε^2:

$$\frac{\partial \rho_{vs}^{*(0)}}{\partial t^*} - \mathrm{div}_{y^*}(D_v^*(\mathbf{grad}_{y^*} \rho_v^{*(2)} + \mathbf{grad}_{x^*} \rho_v^{*(1)})) - \mathrm{div}_{x^*}(D_v^*(\mathbf{grad}_{y^*} \rho_v^{*(1)} + \mathbf{grad}_{x^*} \rho_{vs}^{*(0)}(T^{*(0)}))) = 0 \quad \text{in } \Omega_i \tag{S.B.42}$$

$$D_v^*(\mathbf{grad}_{y^*}\rho_v^{*(2)} + \mathbf{grad}_{x^*}\rho_v^{*(1)}) \cdot \mathbf{n_i} = \rho_i^* w_n^{*(1)} \quad \text{on } \Gamma \tag{S.B.43}$$

where the unknowns $\rho_v^{*(2)}(\mathbf{x}^*, \mathbf{y}^*, t)$ is $\mathbf{y}^*$-periodic and $w_n^{*(1)}$ is the normal interface velocity due to the sublimation/deposition process at the first order. Consequently, integrating (S.B.42) over Ω_a, and then using the divergence theorem, the periodicity condition, and the boundary conditions (S.B.43) leads to the first order dimensionless description:

$$\phi \frac{\partial \rho_{vs}^{*(0)}}{\partial t} - \operatorname{div}_{x^*}(\mathbf{D}^{\mathrm{eff}*} \mathbf{grad}_{x^*} \rho_{vs}^{*(0)}(T^{*(0)})) = \int_\Gamma \rho_i^* w_n^{*(1)} dS = \rho_i^* \dot{\phi} \tag{S.B.44}$$

where $\mathbf{D}^{\mathrm{eff}*}$ is the classical dimensionless effective diffusion tensor defined as (see Case A):

$$\mathbf{D}^{\mathrm{eff}*} = \frac{1}{|\Omega|} \int_{\Omega_a} D_v^*(\mathbf{grad}_{y^*}\, \mathbf{g}_v^*(\mathbf{y}^*) + \mathbf{I}) \mathrm{d}\,\Omega \tag{S.B.45}$$

S.3 Case C

Taking into account the order of magnitude of the dimensionless numbers, $\left[\mathrm{F}_i^T\right] = \mathcal{O}\left(\left[\mathrm{F}_a^T\right]\right) = \mathcal{O}\left([\mathrm{F}_a^\rho]\right) = \mathcal{O}(\varepsilon^2)$, $[\mathrm{K}] = \mathcal{O}(1), [\mathrm{H}] = \mathcal{O}(1), [\mathrm{W_R}] = \mathcal{O}(1)$, the dimensionless microscopic description (13)-(18) becomes:

$$\varepsilon^2 \rho_i^* C_i^* \frac{\partial T_i^*}{\partial t^*} - \operatorname{div}^*(k_i^* \mathbf{grad}^* T_i^*) = 0 \quad \text{in } \Omega_i \tag{S.C.1}$$

$$\varepsilon^2 \rho_a^* C_a^* \frac{\partial T_a^*}{\partial t^*} - \operatorname{div}^*(k_a^* \mathbf{grad} T_a^*) = 0 \quad \text{in } \Omega_a \tag{S.C.2}$$

$$\varepsilon^2 \frac{\partial \rho_v^*}{\partial t^*} - \operatorname{div}^*(D_v^* \mathbf{grad}^* \rho_v^*) = 0 \quad \text{in } \Omega_a \tag{S.C.3}$$

$$T_i^* = T_a^* \quad \text{on } \Gamma \tag{S.C.4}$$

$$k_i^* \mathbf{grad}^* T_i^* \cdot \mathbf{n_i} - k_a^* \mathbf{grad}^* T_a^* \cdot \mathbf{n_i} = L_{sg}^* \frac{D_v^*}{\rho_i^*} \mathbf{grad}^* \rho_v^* \cdot \mathbf{n_i} \quad \text{on } \Gamma \tag{S.C.5}$$

$$D_v^* \mathbf{grad}^* \rho_v^* \cdot \mathbf{n_i} = \rho_i^* \mathbf{w}^* \cdot \mathbf{n_i} \quad \text{on } \Gamma \tag{S.C.6}$$

This set of equations is completed by the Hertz-Knudsen equation (10) and the Clausius Clapeyron's law (9) expressed in dimensionless form as:

$$w_n^* = \mathbf{w}^* \cdot \mathbf{n_i} = \frac{\alpha^*}{\rho_i^*} w_{\mathrm{k}}^* \left[\rho_v^* - \rho_{vs}^*(T_a^*)\right] \quad \text{on } \Gamma \tag{S.C.7}$$

$$\rho_{vs}^*(T_a^*) = \rho_{vs}^{\mathrm{ref}*}(T^{\mathrm{ref}*}) \exp\left[\frac{L_{sg}^* m^*}{\rho_i^* k^*} \left(\frac{1}{T^{\mathrm{ref}*}} - \frac{1}{T_a^*}\right)\right] \tag{S.C.8}$$

S.3.1 Heat and water vapor transfer at the first order

Introducing asymptotic expansions for T_i^* and T_a^* in the relations (S.C.1), (S.C.2), (S.C.4), and (S.C.5) gives at the lowest order:

$$\mathrm{div}_{y^*}(k_i^* \mathbf{grad}_{y^*} T_i^{*(0)}) = 0 \quad \text{in } \Omega_i \tag{S.C.9}$$

$$\mathrm{div}_{y^*}(k_a^* \mathbf{grad}_{y^*} T_a^{*(0)}) = 0 \quad \text{in } \Omega_a \tag{S.C.10}$$

$$T_i^{*(0)} = T_a^{*(0)} \quad \text{on } \Gamma \tag{S.C.11}$$

$$(k_i^* \mathbf{grad}_{y^*}^* T_i^{*(0)} - k_a^* \mathbf{grad}_{y^*}^* T_a^{*(0)}) \cdot \mathbf{n_i} = \frac{L_{sg}^*}{\rho_i^*} D_v^* \mathbf{grad}_{y^*} \rho_v^{*(0)} \cdot \mathbf{n_i} \quad \text{on } \Gamma \tag{S.C.12}$$

where the unknowns $T_i^{*(0)}(\mathbf{x}^*, \mathbf{y}^*, t)$ and $T_a^{*(0)}(\mathbf{x}^*, \mathbf{y}^*, t)$ are $\mathbf{y}^*$-periodic. Introducing asymptotic expansions for ρ_v^* in the relations (S.C.3) and (S.C.6) gives at the lowest order:

$$\mathrm{div}_{y^*}(D_v^* \mathbf{grad}_{y^*} \rho_v^{*(0)}) = 0 \quad \text{in } \Omega_a \tag{S.C.13}$$

$$D_v^* \mathbf{grad}_{y^*} \rho_v^{*(0)} \cdot \mathbf{n_i} = \alpha^* w_\mathrm{k}^* \left[\rho_v^{*(0)} - \rho_{vs}^{*(0)}(T^{*(0)}) \right] \quad \text{on } \Gamma \tag{S.C.14}$$

where the unknown $\rho_v^{*(0)}(\mathbf{x}^*, \mathbf{y}^*, t)$ is $\mathbf{y}^*$-periodic. Consequently, integrating (S.C.13) over Ω_a, and then using the divergence theorem, the periodicity condition, the interface conditions (S.C.14) leads to:

$$\int_\Gamma (\rho_v^{*(0)} - \rho_{vs}^{*(0)}) \mathrm{d}\Gamma = 0 \tag{S.C.15}$$

Taking this result into account, the solution of the above boundary value problem is given by:

$$T_i^{*(0)} = T_a^{*(0)} = T^{*(0)}(\mathbf{x}^*, t) \tag{S.C.16}$$

and

$$\rho_v^{*(0)} = \rho_v^{*(0)}(\mathbf{x}^*, t) = \rho_{vs}^{*(0)}(T^{*(0)}) \tag{S.C.17}$$

At the first order, the temperature and the the water vapor density are independent of the microscopic dimensionless variable $\mathbf{y}^*$, i.e. we have only one temperature field.

S.3.2 Heat and water vapor transfer at the second order

Taking these results into account, Eq. (S.C.1), (S.C.2), (S.C.4), and (S.C.5) of order ε give the following second-order problem:

$$\mathrm{div}_{y^*}(k_i^* (\mathbf{grad}_{y^*} T_i^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)})) = 0 \quad \text{in } \Omega_i \tag{S.C.18}$$

$$\text{div}_{y^*}(k_a^*(\mathbf{grad}_{y^*}T_a^{*(1)} + \mathbf{grad}_{x^*}T^{*(0)})) = 0 \quad \text{in } \Omega_a \tag{S.C.19}$$

$$T_i^{*(1)} = T_a^{*(1)} \quad \text{on } \Gamma \tag{S.C.20}$$

$$(k_i^*(\mathbf{grad}_{y^*}T_i^{*(1)} + \mathbf{grad}_{x^*}T^{*(0)}) - k_a^*(\mathbf{grad}_{y^*}T_a^{*(1)} + \mathbf{grad}_{x^*}T^{*(0)})) \cdot \mathbf{n_i} = \tag{S.C.21}$$

$$\frac{L_{sg}^*}{\rho_i^*} D_v^*(\mathbf{grad}_{y^*}\rho_v^{*(1)} + \mathbf{grad}_{x^*}\rho_v^{*(0)}) \cdot \mathbf{n_i} \quad \text{on } \Gamma$$

where the unknowns $T_i^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t)$ and $T_a^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t)$ are $\mathbf{y}^*$-periodic and the macroscopic gradient $\mathbf{grad}_{x^*}T^{*(0)}$ is given. The second-order problem for the water vapor is given by Eq. (S.C.3) and (S.C.6) of order ε:

$$\text{div}_{y^*}(D_v^*(\mathbf{grad}_{y^*}\rho_v^{*(1)} + \mathbf{grad}_{x^*}\rho_v^{*(0)})) = 0 \quad \text{in } \Omega_a \tag{S.C.22}$$

$$D_v^*(\mathbf{grad}_{y^*}\rho_v^{*(1)} + \mathbf{grad}_{x^*}\rho_v^{*(0)}) \cdot \mathbf{n_i} = \alpha^* w_{\text{k}}^* \left[\rho_v^{*(1)} - \rho_{vs}^{*(1)}\right] \quad \text{on } \Gamma \tag{S.C.23}$$

where the unknowns $\rho_v^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t)$ is $\mathbf{y}^*$-periodic. Consequently, integrating (S.C.22) over Ω_a, and then using the divergence theorem, the periodicity condition, the boundary conditions (S.C.23) leads to:

$$\int_\Gamma (\rho_v^{*(1)} - \rho_{vs}^{*(1)}) \text{d}\Gamma = 0 \tag{S.C.24}$$

The solution of the above boundary value problem for the temperature appears as a linear function of the macroscopic gradient, modulo an arbitrary function $\widetilde{T}^{*(1)}(\mathbf{x}^*, t)$:

$$T_i^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t) = \mathbf{s}_i^*(\mathbf{y}^*) \cdot \mathbf{grad}_{x^*}T^{*(0)} + \widetilde{T}_i^{*(1)} \tag{S.C.25}$$

$$T_a^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t) = \mathbf{s}_a^*(\mathbf{y}^*) \cdot \mathbf{grad}_{x^*}T^{*(0)} + \widetilde{T}_a^{*(1)} \tag{S.C.26}$$

Similarly, we have

$$\rho_v^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t) - \rho_{vs}^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t) = \gamma^*(T^{*(0)})\mathbf{d}^*(\mathbf{y}^*) \cdot \mathbf{grad}_{x^*}T^{*(0)} \tag{S.C.27}$$

with, according to (S.A.40),

$$\rho_{vs}^{*(1)} = \gamma^*(T^{*(0)})\,\mathbf{s}_a^*(\mathbf{y}^*) \cdot \mathbf{grad}_{x^*}T^{*(0)} \tag{S.C.28}$$

Thus,

$$\rho_v^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t) = \gamma^*(T^{*(0)})(\mathbf{d}^*(\mathbf{y}^*) + \mathbf{s}_a^*(\mathbf{y}^*)) \cdot \mathbf{grad}_{x^*}T^{*(0)} \tag{S.C.29}$$

where $\mathbf{s}_i^*(\mathbf{y}^*)$, $\mathbf{s}_a^*(\mathbf{y}^*)$ and $\mathbf{d}^*(\mathbf{y}^*)$ are periodic vectors which characterize the fluctuation of temperature and the water vapor at the pore scale. Introducing (S.C.29), (S.C.26) and (S.C.29) in the set (S.C.18)-(S.C.23), these vectors are solution of the following boundary value problem in a compact form:

$$\text{div}_{y^*}(k_i^*(\mathbf{grad}_{y^*}\mathbf{s}_i^* + \mathbf{I})) = 0 \quad \text{in } \Omega_i \tag{S.C.30}$$

$$\mathrm{div}_{y^*}(k_a^*(\mathbf{grad}_{y^*}\mathbf{s}_a^*+\mathbf{I}))=0 \quad \text{in } \Omega_a \tag{S.C.31}$$

$$\mathbf{s}_i^*=\mathbf{s}_a^* \quad \text{on } \Gamma \tag{S.C.32}$$

$$(k_i^*(\mathbf{grad}_{y^*}\mathbf{s}_i^*+\mathbf{I})-k_a^*(\mathbf{grad}_{y^*}\mathbf{s}_a^*+\mathbf{I}))\cdot\mathbf{n_i}=\frac{L_{sg}^*}{\rho_i^*}\alpha^* w_{\mathrm{k}}^*\gamma^*(T^{*(0)})\mathbf{d}^* \quad \text{on } \Gamma \tag{S.C.33}$$

$$\mathrm{div}_{y^*}(D_v^*(\mathbf{grad}_{y*}(\mathbf{d}^*+\mathbf{s}_{\mathrm{a}}^*)+\mathbf{I}))=0 \quad \text{in } \Omega_a \tag{S.C.34}$$

$$D_v^*(\mathbf{grad}_{y*}(\mathbf{d}^*+\mathbf{s}_{\mathrm{a}}^*)+\mathbf{I})\cdot\mathbf{n_i}=\alpha^* w_{\mathrm{k}}^*\mathbf{d}^* \quad \text{on } \Gamma \tag{S.C.35}$$

with

$$\frac{1}{|\Omega|}\int_\Omega(\mathbf{s}_{\mathrm{a}}^*+\mathbf{s}_{\mathrm{i}}^*)\mathrm{d}\Omega=\mathbf{0} \tag{S.C.36}$$

$$\frac{1}{|\Omega|}\int_\Gamma \mathbf{d}^*\mathrm{d}\Gamma=\mathbf{0} \tag{S.C.37}$$

to ensure the unicity of the solution. Let us remark that the vectors $\mathbf{s}_i^*(\mathbf{y}^*)$, $\mathbf{s}_a^*(\mathbf{y}^*)$ and $\mathbf{d}^*(\mathbf{y}^*)$ depend on the value α. This model is valid for $[\mathrm{W_R}]=\mathcal{O}(1)$, i.e $\varepsilon^{1/2}<[\mathrm{W_R}]<\varepsilon^{-1/2}$. This implies that $(\varepsilon^{1/2}D_{vc}/(lw_{kc}))=\alpha_{\min}<\alpha<\alpha_{\max}=(\varepsilon^{-1/2}D_{vc}/(lw_{kc}))$.

S.3.3 Macroscopic description

Finally, the third order problem for the heat transfer is given by Eq. (S.C.1), (S.C.2), (S.C.4), and (S.C.5) of order ε^2:

$$\rho_i^*C_i^*\frac{\partial T^{*(0)}}{\partial t^*}-\mathrm{div}_{y^*}(k_i^*(\mathbf{grad}_{y^*}T_i^{*(2)}+\mathbf{grad}_{x^*}T_i^{*(1)}))-\mathrm{div}_{x^*}(k_i^*(\mathbf{grad}_{y^*}T_i^{*(1)}+\mathbf{grad}_{x^*}T^{*(0)}))=0 \quad \text{in } \Omega_i \tag{S.C.38}$$

$$\rho_a^*C_a^*\frac{\partial T^{*(0)}}{\partial t^*}-\mathrm{div}_{y^*}(k_a^*(\mathbf{grad}_{y^*}T_a^{*(2)}+\mathbf{grad}_{x^*}T_a^{*(1)}))-\mathrm{div}_{x^*}(k_a^*(\mathbf{grad}_{y^*}T_a^{*(1)}+\mathbf{grad}_{x^*}T^{*(0)}))=0 \quad \text{in } \Omega_a \tag{S.C.39}$$

$$T_i^{*(2)}=T_a^{*(2)} \quad \text{on } \Gamma \tag{S.C.40}$$

$$(k_i^*(\mathbf{grad}_{y^*}T_i^{*(2)}+\mathbf{grad}_{x^*}T_i^{*(1)})-k_a^*(\mathbf{grad}_{y^*}T_a^{*(2)}+\mathbf{grad}_{x^*}T_a^{*(1)}))\cdot\mathbf{n_i}= \\ L_{sg}^*\frac{D_v^*}{\rho_i^*}(\mathbf{grad}_{y^*}\rho_v^{*(2)}+\mathbf{grad}_{x^*}\rho_v^{*(1)}))\cdot\mathbf{n_i} \quad \text{on } \Gamma \tag{S.C.41}$$

where the unknowns $T_i^{*(2)}(\mathbf{x}^*,\mathbf{y}^*,t)$ and $T_a^{*(2)}(\mathbf{x}^*,\mathbf{y}^*,t)$ are $\mathbf{y}^*$-periodic. Integrating (S.C.38) over Ω_i and (S.C.39) over Ω_a, and then using the divergence theorem, the periodicity condition, the interface conditions (S.C.41) and the results leads to the first order dimensionless description:

$$(\rho C)^{\text{eff}*}\frac{\partial T^{*(0)}}{\partial t^*} - \text{div}_{x^*}(\mathbf{k}^{\text{C}*}\mathbf{grad}_{x^*}T^{*(0)}) = -\int_\Gamma L_{sg}^* w_n^{*(2)} dS = L_{sg}^*\dot{\phi} \tag{S.C.42}$$

where $(\rho C)^{\text{eff}*}$ and $\mathbf{k}^{\text{C}*}$ are the dimensionless effective thermal capacity and the effective dimensionless thermal conductivity respectively, defined as:

$$(\rho C)^{\text{eff}*} = (1-\phi)\rho_i^* C_i^* + \phi\rho_a^* C_a^* \tag{S.C.43}$$

$$\mathbf{k}^{\text{C}*} = \frac{1}{|\Omega|}\left(\int_{\Omega_\text{a}} k_a^*(\mathbf{grad}_{y^*}\mathbf{s}_a^*(\mathbf{y}^*)+\mathbf{I})\text{d}\Omega + \int_{\Omega_\text{i}} k_i^*(\mathbf{grad}_{y^*}\mathbf{s}_i^*(\mathbf{y}^*)+\mathbf{I})\text{d}\Omega\right) \tag{S.C.44}$$

where ϕ is the porosity.

Finally, the third order problem for the water vapor is given by Eq. (S.C.3) and (S.C.6) of order ε^2:

$$\frac{\partial \rho_{vs}^{*(0)}}{\partial t^*} - \text{div}_{y^*}(D_v^*(\mathbf{grad}_{y^*}\rho_v^{*(2)} + \mathbf{grad}_{x^*}\rho_v^{*(1)})) - \text{div}_{x^*}(D_v^*(\mathbf{grad}_{y^*}\rho_v^{*(1)} + \mathbf{grad}_{x^*}\rho_{vs}^{*(0)}(T^{*(0)}))) = 0 \quad \text{in } \Omega_i \tag{S.C.45}$$

$$D_v^*(\mathbf{grad}_{y^*}\rho_v^{*(2)} + \mathbf{grad}_{x^*}\rho_v^{*(1)})\cdot\mathbf{n_i} = \rho_i^* w_n^{*(2)} \quad \text{on } \Gamma \tag{S.C.46}$$

where the unknown $\rho_v^{*(2)}(\mathbf{x}^*,\mathbf{y}^*,t)$ is $\mathbf{y}^*$-periodic and $w_n^{*(2)}$ is the normal interface velocity due to the sublimation/deposition process at the first order. Consequently, integrating (S.C.45) over Ω_a, and then using the divergence theorem, the periodicity condition, and the interface conditions (S.C.46) leads to the first order dimensionless description:

$$\phi\frac{\partial \rho_{vs}^{*(0)}}{\partial t} - \text{div}_{x^*}(\mathbf{D}^{\text{C}*}\mathbf{grad}_{x^*}\rho_{vs}^{*(0)}(T^{*(0)})) = \int_\Gamma \rho_i^* w_n^{*(2)} dS = \rho_i^*\dot{\phi} \tag{S.C.47}$$

where $\mathbf{D}^{\text{C}*}$ is the classical dimensionless effective diffusion tensor defined as:

$$\mathbf{D}^{\text{C}*} = \frac{1}{|\Omega|}\int_{\Omega_a} D_v^*(\mathbf{grad}_{y^*}\ (\mathbf{d}^* + \mathbf{s}_\text{a}^*) + \mathbf{I})\text{d}\Omega \tag{S.C.48}$$

S.4 Cases D1 and D2

S.4.1 Case D1

Taking into account the order of magnitude of the dimensionless numbers, $\left[\text{F}_i^T\right] = \mathcal{O}\left(\left[\text{F}_a^T\right]\right) = \mathcal{O}\left([\text{F}_a^\rho]\right) = \mathcal{O}(\varepsilon^2)$, $[\text{K}] = \mathcal{O}(1)$, $[\text{W}_\text{R}] = \mathcal{O}\left(\varepsilon^{-1}\right)$, $[\text{H}] = \mathcal{O}(1)$, the dimensionless microscopic description (13)-(18) becomes:

$$\varepsilon^2\rho_i^* C_i^*\frac{\partial T_i^*}{\partial t^*} - \text{div}^*(k_i^*\mathbf{grad}^* T_i^*) = 0 \quad \text{in } \Omega_i \tag{S.D1.1}$$

$$\varepsilon^2 \rho_a^* C_a^* \frac{\partial T_a^*}{\partial t^*} - \mathrm{div}^*(k_a^* \mathbf{grad} T_a^*) = 0 \quad \text{in } \Omega_a \tag{S.D1.2}$$

$$\varepsilon^2 \frac{\partial \rho_v^*}{\partial t^*} - \mathrm{div}^*(D_v^* \mathbf{grad}^* \rho_v^*) = 0 \quad \text{in } \Omega_a \tag{S.D1.3}$$

$$T_i^* = T_a^* \quad \text{on } \Gamma \tag{S.D1.4}$$

$$k_i^* \mathbf{grad}^* T_i^* \cdot \mathbf{n_i} - k_a^* \mathbf{grad}^* T_a^* \cdot \mathbf{n_i} = L_{sg}^* \frac{D_v^*}{\rho_i^*} \mathbf{grad}^* \rho_v^* \cdot \mathbf{n_i} \quad \text{on } \Gamma \tag{S.D1.5}$$

$$D_v^* \mathbf{grad}^* \rho_v^* \cdot \mathbf{n_i} = \varepsilon^{-1} \rho_i^* \mathbf{w}^* \cdot \mathbf{n_i} \quad \text{on } \Gamma \tag{S.D1.6}$$

This set of equations is completed by the Hertz-Knudsen equation (10) and the Clausius Clapeyron's law (9) expressed in dimensionless form as:

$$w_n^* = \mathbf{w}^* \cdot \mathbf{n_i} = \frac{\alpha^*}{\rho_i^*} w_{\mathrm{k}}^* \left[\rho_v^* - \rho_{vs}^*(T_a^*)\right] \quad \text{on } \Gamma \tag{S.D1.7}$$

$$\rho_{vs}^*(T_a^*) = \rho_{vs}^{\mathrm{ref}*}(T^{\mathrm{ref}*}) \exp\left[\frac{L_{sg}^* m^*}{\rho_i^* k^*} \left(\frac{1}{T^{\mathrm{ref}*}} - \frac{1}{T_a^*}\right)\right] \tag{S.D1.8}$$

S.4.1.1 Heat transfer and water vapor transfer at the first and the second order

Introducing asymptotic expansions for T_i^* and T_a^* in the relations (S.D1.1), (S.D1.2), (S.D1.4), and (S.D1.5) gives at the lowest order:

$$\mathrm{div}_{y^*}(k_i^* \mathbf{grad}_{y^*} T_i^{*(0)}) = 0 \quad \text{in } \Omega_i \tag{S.D1.9}$$

$$\mathrm{div}_{y^*}(k_a^* \mathbf{grad}_{y^*} T_a^{*(0)}) = 0 \quad \text{in } \Omega_a \tag{S.D1.10}$$

$$T_i^{*(0)} = T_a^{*(0)} \quad \text{on } \Gamma \tag{S.D1.11}$$

$$(k_i^* \mathbf{grad}_{y^*}^* T_i^{*(0)} - k_a^* \mathbf{grad}_{y^*}^* T_a^{*(0)}) \cdot \mathbf{n_i} = L_{sg}^* \frac{D_v^*}{\rho_i^*} \mathbf{grad}_{y^*} \rho_v^{*(0)} \quad \text{on } \Gamma \tag{S.D1.12}$$

where the unknowns $T_i^{*(0)}(\mathbf{x}^*, \mathbf{y}^*, t)$ and $T_a^{*(0)}(\mathbf{x}^*, \mathbf{y}^*, t)$ are $\mathbf{y}^*$-periodic. Introducing asymptotic expansions for ρ_v^* in the relations (S.D1.3, S.D1.6) gives at the lowest order:

$$\mathrm{div}_{y^*}(D_v^* \mathbf{grad}_{y^*} \rho_v^{*(0)}) = 0 \quad \text{in } \Omega_a \tag{S.D1.13}$$

$$\rho_v^{*(0)} = \rho_{vs}^{*(0)}(T^{*(0)}) \quad \text{on } \Gamma \tag{S.D1.14}$$

where the unknown $\rho_v^{*(0)}(\mathbf{x}^*, \mathbf{y}^*, t)$ is $\mathbf{y}^*$-periodic. The solution of the above boundary value problems is given by:

$$\rho_v^{*(0)} = \rho_v^{*(0)}(\mathbf{x}^*, t) = \rho_{vs}^{*(0)}(T^{*(0)}) \tag{S.D1.15}$$

$$T_i^{*(0)} = T_a^{*(0)} = T^{*(0)}(\mathbf{x}^*, t) \tag{S.D1.16}$$

At the first order, the temperature and the water vapor density are independent of the microscopic dimensionless variable $\mathbf{y}^*$. We have only one temperature field. Taking these results into account, Eq. (S.D1.1), (S.D1.2), (S.D1.4), and (S.D1.5) of order ε give the following second-order problem:

$$\text{div}_{y^*}(k_i^*(\mathbf{grad}_{y^*} T_i^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)})) = 0 \quad \text{in } \Omega_i \tag{S.D1.17}$$

$$\text{div}_{y^*}(k_a^*(\mathbf{grad}_{y^*} T_a^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)})) = 0 \quad \text{in } \Omega_a \tag{S.D1.18}$$

$$T_i^{*(1)} = T_a^{*(1)} \quad \text{on } \Gamma \tag{S.D1.19}$$

$$(k_i^*(\mathbf{grad}_{y^*} T_i^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)}) - k_a^*(\mathbf{grad}_{y^*} T_a^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)})) \cdot \mathbf{n_i} = L_{sg}^* \frac{D_v^*}{\rho_i^*}(\mathbf{grad}_{y^*} \rho_v^{*(1)} + \mathbf{grad}_{x^*} \rho_v^{*(0)}) \cdot \mathbf{n_i} \quad \text{on } \Gamma \tag{S.D1.20}$$

where the unknowns $T_i^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t)$ and $T_a^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t)$ are $\mathbf{y}^*$-periodic and the macroscopic gradient $\mathbf{grad}_{x^*} T^{*(0)}$ is given. Moreover we have the second-order problem for Eq. (S.D1.3) and (S.D1.6) is written:

$$\text{div}_{y^*}(D_v^*(\mathbf{grad}_{y^*} \rho_v^{*(1)} + \mathbf{grad}_{x^*} \rho_{vs}^{*(0)})) = 0 \quad \text{in } \Omega_a \tag{S.D1.21}$$

$$\rho_v^{*(1)} = \rho_{vs}^{*(1)} \quad \text{on } \Gamma \tag{S.D1.22}$$

where the unknowns $\rho_v^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t)$ is $\mathbf{y}^*$-periodic. According to (S.A.40), this latter boundary condition can be also written

$$\rho_v^{*(1)} = \rho_{vs}^{*(1)} = \gamma^*(T^{*(0)}) T_a^{*(1)} \quad \text{on } \Gamma \tag{S.D1.23}$$

Moreover, we have

$$\mathbf{grad}_{x^*} \rho_{vs}^{*(0)} = \gamma^*(T^{*(0)}) \mathbf{grad}_{x^*} T^{*(0)} \tag{S.D1.24}$$

thus Eq. (S.D1.21) and (S.D1.23) are written:

$$\text{div}_{y^*}(D_v^*(\mathbf{grad}_{y^*} \rho_v^{*(1)} + \gamma^*(T^{*(0)}) \mathbf{grad}_{x^*} T^{*(0)}) = 0 \quad \text{in } \Omega_a \tag{S.D1.25}$$

$$\rho_v^{*(1)} = \gamma^*(T^{*(0)})T_a^{*(1)} \quad \text{on } \Gamma \tag{S.D1.26}$$

The solution of the above boundary value problems (S.D1.17)-(S.D1.20) and (S.D1.25)-(S.D1.26) appears as a linear function of the macroscopic gradient $\mathbf{grad}_{x^*}T^{*(0)}$, modulo an arbitrary function.

$$T_i^{*(1)}(\mathbf{x}^*,\mathbf{y}^*,t) = \mathbf{r}_i^*(\mathbf{y}^*)\cdot\mathbf{grad}_{x^*}T^{*(0)} + \widetilde{T}_i^{*(1)} \tag{S.D1.27}$$

$$T_a^{*(1)}(\mathbf{x}^*,\mathbf{y}^*,t) = \mathbf{r}_a^*(\mathbf{y}^*)\cdot\mathbf{grad}_{x^*}T^{*(0)} + \widetilde{T}_a^{*(1)} \tag{S.D1.28}$$

$$\rho_v^{*(1)}(\mathbf{x}^*,\mathbf{y}^*,t) = \gamma^*(T^{*(0)})\mathbf{r}_a^*(\mathbf{y}^*)\cdot\mathbf{grad}_{x^*}T^{*(0)} + \widetilde{T}_a^{*(1)} \quad \text{on } \Gamma \tag{S.D1.29}$$

where $\mathbf{r}_i^*(\mathbf{y}^*)$ and $\mathbf{r}_a^*(\mathbf{y}^*)$ are two periodic vectors which characterize the fluctuation of temperature in both phases at the pore scale. Introducing (S.D1.27) and (S.D1.28) in the set (S.D1.17)-(S.D1.20), these two vectors are solution of the following boundary value problem in a compact form:

$$\mathrm{div}_{y^*}(k_i^*(\mathbf{grad}_{y^*}\mathbf{r}_i^* + \mathbf{I})) = 0 \quad \text{in } \Omega_i \tag{S.D1.30}$$

$$\mathrm{div}_{y^*}((k_a^* + L_{sg}^* D_v^* \frac{\gamma^*(T^{*(0)})}{\rho_i^*})(\mathbf{grad}_{y^*}\mathbf{r}_a^* + \mathbf{I})) = 0 \quad \text{in } \Omega_a \tag{S.D1.31}$$

$$\mathbf{r}_i^* = \mathbf{r}_a^* \quad \text{on } \Gamma \tag{S.D1.32}$$

$$(k_i^*(\mathbf{grad}_{y^*}\mathbf{r}_i^* + \mathbf{I}) - (k_a^* + L_{sg}^* D_v^* \frac{\gamma^*(T^{*(0)})}{\rho_i^*})(\mathbf{grad}_{y^*}\mathbf{r}_a^* + \mathbf{I}))\cdot\mathbf{n_i} = 0 \quad \text{on } \Gamma \tag{S.D1.33}$$

$$\frac{1}{|\Omega|}\int_\Omega (\mathbf{r}_\mathrm{a}^* + \mathbf{r}_\mathrm{i}^*)\mathrm{d}\Omega = \mathbf{0} \tag{S.D1.34}$$

This latter equation is introduced to ensure the uniqueness of the solution. This latter boundary value problem is similar to the one of the Eq. (S.A.20)-(S.A.24) where k_a^* is now equal to $k_a^* + L_{sg}^* D_v^* \gamma^*(T^{*(0)})/\rho_i^*$. At the local scale, the thermal conductivity appears to be enhanced by the phase change.

S.4.1.2 Macroscopic description

Finally, the third order problem is given by the equations (S.D1.1, S.D1.2, S.D1.4, S.D1.5) of order ε^2:

$$\rho_i^* C_i^* \frac{\partial T^{*(0)}}{\partial t^*} - \mathrm{div}_{y^*}(k_i^*(\mathbf{grad}_{y^*}T_i^{*(2)} + \mathbf{grad}_{x^*}T_i^{*(1)})) - \mathrm{div}_{x^*}(k_i^*(\mathbf{grad}_{y^*}T_i^{*(1)} + \mathbf{grad}_{x^*}T^{*(0)})) = 0 \quad \text{in } \Omega_i \tag{S.D1.35}$$

$$\rho_a^* C_a^* \frac{\partial T^{*(0)}}{\partial t^*} - \mathrm{div}_{y^*}(k_a^*(\mathbf{grad}_{y^*}T_a^{*(2)} + \mathbf{grad}_{x^*}T_a^{*(1)})) - \mathrm{div}_{x^*}(k_a^*(\mathbf{grad}_{y^*}T_a^{*(1)} + \mathbf{grad}_{x^*}T^{*(0)})) = 0 \quad \text{in } \Omega_a \tag{S.D1.36}$$

$$T_i^{*(2)} = T_a^{*(2)} \quad \text{on } \Gamma \tag{S.D1.37}$$

$$(k_i^*(\mathbf{grad}_{y^*} T_i^{*(2)} + \mathbf{grad}_{x^*} T_i^{*(1)}) - k_a^*(\mathbf{grad}_{y^*} T_a^{*(2)} + \mathbf{grad}_{x^*} T_a^{*(1)})) \cdot \mathbf{n_i} = L_{sg}^* \frac{D_v^*}{\rho_i^*}(\mathbf{grad}_{y^*} \rho_v^{*(2)} + \mathbf{grad}_{x^*} \rho_v^{*(1)}) \cdot \mathbf{n_i} \quad \text{on } \Gamma \tag{S.D1.38}$$

where the unknowns $T_i^{*(2)}(\mathbf{x}^*, \mathbf{y}^*, t)$ and $T_a^{*(2)}(\mathbf{x}^*, \mathbf{y}^*, t)$ are $\mathbf{y}^*$-periodic. For the water vapor, the third order problem is given by the the equations (S.D1.3, S.D1.6) of order ε^2:

$$\frac{\partial \rho_{vs}^{*(0)}}{\partial t^*} - \text{div}_{y^*}(D_v^*(\mathbf{grad}_{y^*} \rho_v^{*(2)} + \mathbf{grad}_{x^*} \rho_v^{*(1)})) - \text{div}_{x^*}(D_v^*(\mathbf{grad}_{y^*} \rho_v^{*(1)} + \mathbf{grad}_{x^*} \rho_{vs}^{*(0)}(T^{*(0)}))) = 0 \quad \text{in } \Omega_a \tag{S.D1.39}$$

$$D_v^*(\mathbf{grad}_{y^*} \rho_v^{*(2)} + \mathbf{grad}_{x^*} \rho_v^{*(1)}) \cdot \mathbf{n_i} = \rho_i^* w_n^{*(3)} \quad \text{on } \Gamma \tag{S.D1.40}$$

Integrating (S.D1.35) over Ω_i and (S.D1.36) and (S.D1.39) over Ω_a, and then using the divergence theorem, the periodicity condition, and the interface conditions (S.C.41) leads to the first order dimensionless description:

$$(\rho C)^{\text{eff}*} \frac{\partial T^{*(0)}}{\partial t^*} - \text{div}_{x^*}(\mathbf{k}^{\text{D}*} \mathbf{grad}_{x^*} T^{*(0)}) = \int_\Gamma L_{sg}^* \frac{D_v^*}{\rho_i^*}(\mathbf{grad}_{y^*} \rho_v^{*(2)} + \mathbf{grad}_{x^*} \rho_v^{*(1)}) \cdot \mathbf{n_i} dS = -L_{sg}^* \dot{\phi} \tag{S.D1.41}$$

where $(\rho C)^{\text{eff}*}$ and $\mathbf{k}^{\text{D}*}$ are the dimensionless effective thermal capacity and the apparent dimensionless conductivity respectively, defined as:

$$(\rho C)^{\text{eff}*} = (1-\phi)\rho_i^* C_i^* + \phi \rho_a^* C_a^* \tag{S.D1.42}$$

$$\mathbf{k}^{\text{D}*} = \frac{1}{|\Omega|}\left(\int_{\Omega_\text{a}} k_a^*(\mathbf{grad}_{y^*} \mathbf{r}_a^*(\mathbf{y}^*) + \mathbf{I}) \text{d}\Omega + \int_{\Omega_\text{i}} k_i^*(\mathbf{grad}_{y^*} \mathbf{r}_i^*(\mathbf{y}^*) + \mathbf{I}) \text{d}\Omega\right) \tag{S.D1.43}$$

where ϕ is the porosity. Integrating (S.D1.39) over Ω_a, and then using the divergence theorem and the periodicity condition, leads to the first order dimensionless description:

$$\phi \frac{\partial \rho_{vs}^{*(0)}}{\partial t} - \text{div}_{x^*}(\mathbf{D}^{\text{D}*} \mathbf{grad}_{x^*} \rho_{vs}^{*(0)}(T^{*(0)})) = -\int_\Gamma D_v^*(\mathbf{grad}_{y^*} \rho_v^{*(2)} + \mathbf{grad}_{x^*} \rho_v^{*(1)}) \cdot \mathbf{n_i} dS = \rho_i^* \dot{\phi} \tag{S.D1.44}$$

where $\mathbf{D}^{\text{D}*}$ is the apparent effective diffusion tensor defined as:

$$\mathbf{D}^{\text{D}*} = \frac{1}{|\Omega|} \int_{\Omega_a} D_v^*(\mathbf{grad}_{y^*} \mathbf{r}_a^*(\mathbf{y}^*) + \mathbf{I}) \text{d}\,\Omega \tag{S.D1.45}$$

S.4.2 Case D2

Taking into account the order of magnitude of the dimensionless numbers, $\left[\mathrm{F}_i^T\right] = \mathcal{O}\left(\left[\mathrm{F}_a^T\right]\right) = \mathcal{O}\left([\mathrm{F}_a^\rho]\right) = \mathcal{O}(\varepsilon^2)$, $[\mathrm{K}] = \mathcal{O}(1)$, $[\mathrm{W_R}] = \mathcal{O}\left(\varepsilon^{-1}\right)$, $[\mathrm{H}] = \mathcal{O}(1)$, the dimensionless microscopic description (13)-(18) becomes:

$$\varepsilon^2 \rho_i^* C_i^* \frac{\partial T_i^*}{\partial t^*} - \mathrm{div}^*(k_i^* \mathbf{grad}^* T_i^*) = 0 \quad \text{in } \Omega_i \tag{S.D2.1}$$

$$\varepsilon^2 \rho_a^* C_a^* \frac{\partial T_a^*}{\partial t^*} - \mathrm{div}^*(k_a^* \mathbf{grad} T_a^*) = 0 \quad \text{in } \Omega_a \tag{S.D2.2}$$

$$\varepsilon^2 \frac{\partial \rho_v^*}{\partial t^*} - \mathrm{div}^*(D_v^* \mathbf{grad}^* \rho_v^*) = 0 \quad \text{in } \Omega_a \tag{S.D2.3}$$

$$T_i^* = T_a^* \quad \text{on } \Gamma \tag{S.D2.4}$$

$$k_i^* \mathbf{grad}^* T_i^* \cdot \mathbf{n_i} - k_a^* \mathbf{grad}^* T_a^* \cdot \mathbf{n_i} = L_{sg}^* \frac{D_v^*}{\rho_i^*} \mathbf{grad}^* \rho_v^* \cdot \mathbf{n_i} \quad \text{on } \Gamma \tag{S.D2.5}$$

$$D_v^* \mathbf{grad}^* \rho_v^* \cdot \mathbf{n_i} = \varepsilon^{-2} \rho_i^* \mathbf{w}^* \cdot \mathbf{n_i} \quad \text{on } \Gamma \tag{S.D2.6}$$

This set of equations is completed by the Hertz-Knudsen equation (S.A.7) and the Clausius Clapeyron's law (9) expressed in dimensionless form as:

$$w_n^* = \mathbf{w}^* \cdot \mathbf{n_i} = \frac{\alpha^*}{\rho_i^*} w_{\mathrm{k}}^* \left[\rho_v^* - \rho_{vs}^*(T_a^*)\right] \quad \text{on } \Gamma \tag{S.D2.7}$$

$$\rho_{vs}^*(T_a^*) = \rho_{vs}^{\mathrm{ref}*}(T^{\mathrm{ref}*}) \exp\left[\frac{L_{sg}^* m^*}{\rho_i^* k^*} \left(\frac{1}{T^{\mathrm{ref}*}} - \frac{1}{T_a^*}\right)\right] \tag{S.D2.8}$$

S.4.2.1 Heat transfer and water vapor transfer at the first and second order

Introducing asymptotic expansions for T_i^* and T_a^* in the relations (S.D2.1, S.D2.2, S.D2.4, S.D2.5) gives at the lowest order:

$$\mathrm{div}_{y^*}(k_i^* \mathbf{grad}_{y^*} T_i^{*(0)}) = 0 \quad \text{in } \Omega_i \tag{S.D2.9}$$

$$\mathrm{div}_{y^*}(k_a^* \mathbf{grad}_{y^*} T_a^{*(0)}) = 0 \quad \text{in } \Omega_a \tag{S.D2.10}$$

$$T_i^{*(0)} = T_a^{*(0)} \quad \text{on } \Gamma \tag{S.D2.11}$$

$$(k_i^* \mathbf{grad}_{y^*}^* T_i^{*(0)} - k_a^* \mathbf{grad}_{y^*}^* T_a^{*(0)}) \cdot \mathbf{n_i} = L_{sg}^* \frac{D_v^*}{\rho_i^*} \mathbf{grad}_{y^*} \rho_v^{*(0)} \quad \text{on } \Gamma \tag{S.D2.12}$$

where the unknowns $T_i^{*(0)}(\mathbf{x}^*, \mathbf{y}^*, t)$ and $T_a^{*(0)}(\mathbf{x}^*, \mathbf{y}^*, t)$ are $\mathbf{y}^*$-periodic. Introducing asymptotic expansions for ρ_v^* in the relations (S.D2.3, S.D2.6) gives at the lowest order

$$\mathrm{div}_{y^*}(D_v^* \mathbf{grad}_{y^*} \rho_v^{*(0)}) = 0 \quad \text{in } \Omega_a \tag{S.D2.13}$$

$$\rho_v^{*(0)} = \rho_{vs}^{*(0)}(T^{*(0)}) \quad \text{on } \Gamma \tag{S.D2.14}$$

where the unknowns $\rho_v^{*(0)}(\mathbf{x}^*, \mathbf{y}^*, t)$ is $\mathbf{y}^*$-periodic. The solution of the above boundary value problems is given by:

$$\rho_v^{*(0)} = \rho_v^{*(0)}(\mathbf{x}^*, t) = \rho_{vs}^{*(0)}(T^{*(0)}) \tag{S.D2.15}$$

$$T_i^{*(0)} = T_a^{*(0)} = T^{*(0)}(\mathbf{x}^*, t) \tag{S.D2.16}$$

At the first order, the temperature and the water vapor density are independent of the microscopic dimensionless variable $\mathbf{y}^*$. We have only one temperature field. Taking these results into account, equations (S.D2.1, S.D2.2, S.D2.4, S.D2.5) of order ε give the following second-order problem:

$$\mathrm{div}_{y^*}(k_i^*(\mathbf{grad}_{y^*} T_i^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)})) = 0 \quad \text{in } \Omega_i \tag{S.D2.17}$$

$$\mathrm{div}_{y^*}(k_a^*(\mathbf{grad}_{y^*} T_a^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)})) = 0 \quad \text{in } \Omega_a \tag{S.D2.18}$$

$$T_i^{*(1)} = T_a^{*(1)} \quad \text{on } \Gamma \tag{S.D2.19}$$

$$(k_i^*(\mathbf{grad}_{y^*} T_i^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)}) - k_a^*(\mathbf{grad}_{y^*} T_a^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)})) \cdot \mathbf{n_i} = \tag{S.D2.20}$$

$$L_{sg}^* \frac{D_v^*}{\rho_i^*}(\mathbf{grad}_{y^*} \rho_v^{*(1)} + \mathbf{grad}_{x^*} \rho_v^{*(0)}) \cdot \mathbf{n_i} \quad \text{on } \Gamma$$

where the unknowns $T_i^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t)$ and $T_a^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t)$ are $\mathbf{y}^*$-periodic and the macroscopic gradient $\mathbf{grad}_{x^*} T^{*(0)}$ is given. Moreover we have the second-order problem for the equations (S.D2.3, S.D2.6) is written:

$$\mathrm{div}_{y^*}(D_v^*(\mathbf{grad}_{y^*} \rho_v^{*(1)} + \mathbf{grad}_{x^*} \rho_{vs}^{*(0)})) = 0 \quad \text{in } \Omega_a \tag{S.D2.21}$$

$$\rho_v^{*(1)} = \rho_{vs}^{*(1)} \quad \text{on } \Gamma \tag{S.D2.22}$$

where the unknowns $\rho_v^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t)$ is $\mathbf{y}^*$-periodic. According to (S.A.40), this latter boundary condition can be also written

$$\rho_v^{*(1)} = \rho_{vs}^{*(1)} = \gamma^*(T^{*(0)}) T_a^{*(1)} \quad \text{on } \Gamma \tag{S.D2.23}$$

Moreover, we have

$$\mathbf{grad}_{x^*} \rho_{vs}^{*(0)} = \gamma^*(T^{*(0)}) \mathbf{grad}_{x^*} T^{*(0)} \tag{S.D2.24}$$

thus equations (S.D2.21) and (S.D2.23) are written:

$$\mathrm{div}_{y^*}(D_v^*(\mathbf{grad}_{y^*}\rho_v^{*(1)} + \gamma^*(T^{*(0)})\mathbf{grad}_{x^*}T^{*(0)}) = 0 \quad \text{in } \Omega_a \tag{S.D2.25}$$

$$\rho_v^{*(1)} = \gamma^*(T^{*(0)})\,T_a^{*(1)} \quad \text{on } \Gamma \tag{S.D2.26}$$

As in the Cases C1 and C2, the solution of the above boundary value problems (S.D2.17-S.D2.20) and (S.D2.25-S.D2.26) appears as a linear function of the macroscopic gradient $\mathbf{grad}_{x^*}T^{*(0)}$, modulo an arbitrary function.

$$T_i^{*(1)}(\mathbf{x}^*,\mathbf{y}^*,t) = \mathbf{r}_i^*(\mathbf{y}^*)\cdot\mathbf{grad}_{x^*}T^{*(0)} + \widetilde{T}_i^{*(1)} \tag{S.D2.27}$$

$$T_a^{*(1)}(\mathbf{x}^*,\mathbf{y}^*,t) = \mathbf{r}_a^*(\mathbf{y}^*)\cdot\mathbf{grad}_{x^*}T^{*(0)} + \widetilde{T}_a^{*(1)} \tag{S.D2.28}$$

$$\rho_v^{*(1)}(\mathbf{x}^*,\mathbf{y}^*,t) = \gamma^*(T^{*(0)})\,(\mathbf{r}_a^*(\mathbf{y}^*)\cdot\mathbf{grad}_{x^*}T^{*(0)} + \widetilde{T}_a^{*(1)}) \quad \text{on } \Gamma \tag{S.D2.29}$$

where $\mathbf{r}_i^*(\mathbf{y}^*)$ and $\mathbf{r}_a^*(\mathbf{y}^*)$ are two periodic vectors which characterize the fluctuation of temperature in both phases at the pore scale. Introducing (S.D2.27) and (S.D2.28) in the set (S.D2.17-S.D2.20), these two vectors are solution of the following boundary value problem in a compact form:

$$\mathrm{div}_{y^*}(k_i^*(\mathbf{grad}_{y^*}\mathbf{r}_i^* + \mathbf{I})) = 0 \quad \text{in } \Omega_i \tag{S.D2.30}$$

$$\mathrm{div}_{y^*}((k_a^* + L_{sg}^* D_v^* \frac{\gamma^*(T^{*(0)})}{\rho_i^*})(\mathbf{grad}_{y^*}\mathbf{r}_a^* + \mathbf{I})) = 0 \quad \text{in } \Omega_a \tag{S.D2.31}$$

$$\mathbf{r}_i^* = \mathbf{r}_a^* \quad \text{on } \Gamma \tag{S.D2.32}$$

$$(k_i^*(\mathbf{grad}_{y^*}\mathbf{r}_i^* + \mathbf{I}) - (k_a^* + L_{sg}^* D_v^* \frac{\gamma^*(T^{*(0)})}{\rho_i^*})(\mathbf{grad}_{y^*}\mathbf{r}_a^* + \mathbf{I}))\cdot\mathbf{n_i} = 0 \quad \text{on } \Gamma \tag{S.D2.33}$$

$$\frac{1}{|\Omega|}\int_{\Omega}(\mathbf{r}_\mathrm{a}^* + \mathbf{r}_\mathrm{i}^*)\mathrm{d}\Omega = \mathbf{0} \tag{S.D2.34}$$

This latter equation is introduced to ensure the uniqueness of the solution. This latter boundary value problem is similar to the one of the Eq. (S.A.20)-(S.A.24) where k_a^* is now equal to $k_a^* + L_{sg}^* D_v^* \gamma^*(T^{*(0)})/\rho_i^*$. At the local scale, the thermal conductivity appears to be enhanced by the phase change.

S.4.2.2 Macroscopic description

Finally, the third order problem is given by Eq. (S.D2.1), (S.D2.2), (S.D2.4), and (S.D2.5) of order ε^2:

$$\rho_i^* C_i^* \frac{\partial T^{*(0)}}{\partial t^*} - \mathrm{div}_{y^*}(k_i^*(\mathbf{grad}_{y^*} T_i^{*(2)} + \mathbf{grad}_{x^*} T_i^{*(1)})) - \mathrm{div}_{x^*}(k_i^*(\mathbf{grad}_{y^*} T_i^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)})) = 0 \quad \text{in } \Omega_i \quad \text{(S.D2.35)}$$

$$\rho_a^* C_a^* \frac{\partial T^{*(0)}}{\partial t^*} - \mathrm{div}_{y^*}(k_a^*(\mathbf{grad}_{y^*} T_a^{*(2)} + \mathbf{grad}_{x^*} T_a^{*(1)})) - \mathrm{div}_{x^*}(k_a^*(\mathbf{grad}_{y^*} T_a^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)})) = 0 \quad \text{in } \Omega_a \quad \text{(S.D2.36)}$$

$$T_i^{*(2)} = T_a^{*(2)} \quad \text{on } \Gamma \qquad \text{(S.D2.37)}$$

$$(k_i^*(\mathbf{grad}_{y^*} T_i^{*(2)} + \mathbf{grad}_{x^*} T_i^{*(1)}) - k_a^*(\mathbf{grad}_{y^*} T_a^{*(2)} + \mathbf{grad}_{x^*} T_a^{*(1)})) \cdot \mathbf{n_i} = L_{sg}^* \frac{D_v^*}{\rho_i^*}(\mathbf{grad}_{y^*} \rho_v^{*(2)} + \mathbf{grad}_{x^*} \rho_v^{*(1)}) \cdot \mathbf{n_i} \quad \text{on } \Gamma \qquad \text{(S.D2.38)}$$

where the unknowns $T_i^{*(2)}(\mathbf{x}^*, \mathbf{y}^*, t)$ and $T_a^{*(2)}(\mathbf{x}^*, \mathbf{y}^*, t)$ are $\mathbf{y}^*$-periodic. For the water vapor, the third order problem is given by Eq. (S.D2.3) and (S.D2.6) of order ε^2:

$$\frac{\partial \rho_{vs}^{*(0)}}{\partial t^*} - \mathrm{div}_{y^*}(D_v^*(\mathbf{grad}_{y^*} \rho_v^{*(2)} + \mathbf{grad}_{x^*} \rho_v^{*(1)})) - \mathrm{div}_{x^*}(D_v^*(\mathbf{grad}_{y^*} \rho_v^{*(1)} + \mathbf{grad}_{x^*} \rho_{vs}^{*(0)}(T^{*(0)}))) = 0 \quad \text{in } \Omega_a \quad \text{(S.D2.39)}$$

$$D_v^*(\mathbf{grad}_{y^*} \rho_v^{*(2)} + \mathbf{grad}_{x^*} \rho_v^{*(1)}) \cdot \mathbf{n_i} = \rho_i^* w_n^{*(4)} \quad \text{on } \Gamma \qquad \text{(S.D2.40)}$$

Integrating (S.D2.35) over Ω_i and (S.D2.36) and (S.D2.39) over Ω_a, and then using the divergence theorem, the periodicity condition, and the interface conditions (S.C.41) leads to the first order dimensionless description:

$$(\rho C)^{\mathrm{eff}*} \frac{\partial T^{*(0)}}{\partial t^*} - \mathrm{div}_{x^*}(\mathbf{k}^{\mathrm{D}*} \mathbf{grad}_{x^*} T^{*(0)}) = \int_\Gamma L_{sg}^* \frac{D_v^*}{\rho_i^*}(\mathbf{grad}_{y^*} \rho_v^{*(2)} + \mathbf{grad}_{x^*} \rho_v^{*(1)}) \cdot \mathbf{n_i} dS = -L_{sg}^* \dot{\phi} \qquad \text{(S.D2.41)}$$

where $(\rho C)^{\mathrm{eff}*}$ and $\mathbf{k}^{\mathrm{D}*}$ are the dimensionless effective thermal capacity and the apparent dimensionless thermal conductivity, respectively, defined as:

$$(\rho C)^{\mathrm{eff}*} = (1-\phi)\rho_i^* C_i^* + \phi \rho_a^* C_a^* \qquad \text{(S.D2.42)}$$

$$\mathbf{k}^{\mathrm{D}*} = \frac{1}{|\Omega|}\left(\int_{\Omega_\mathrm{a}} k_a^*(\mathbf{grad}_{y^*} \mathbf{r}_a^*(\mathbf{y}^*) + \mathbf{I}) \mathrm{d}\Omega + \int_{\Omega_\mathrm{i}} k_i^*(\mathbf{grad}_{y^*} \mathbf{r}_i^*(\mathbf{y}^*) + \mathbf{I}) \mathrm{d}\Omega\right) \qquad \text{(S.D2.43)}$$

where ϕ is the porosity. Integrating (S.D2.39) over Ω_a, and then using the divergence theorem and the periodicity condition, leads to the first order dimensionless description:

$$\phi \frac{\partial \rho_{vs}^{*(0)}}{\partial t} - \mathrm{div}_{x^*}(\mathbf{D}^{\mathrm{D}*} \mathbf{grad}_{x^*} \rho_{vs}^{*(0)}(T^{*(0)})) = -\int_\Gamma D_v^*(\mathbf{grad}_{y^*} \rho_v^{*(2)} + \mathbf{grad}_{x^*} \rho_v^{*(1)}) \cdot \mathbf{n_i} dS = \rho_i^* \dot{\phi} \qquad \text{(S.D2.44)}$$

where $\mathbf{D}^{\mathrm{D}*}$ is the apparent effective diffusion tensor defined as:

$$\mathbf{D}^{\mathrm{D}*} = \frac{1}{|\Omega|}\int_{\Omega_a} D_v^*(\mathbf{grad}_{y^*}\, \mathbf{r}_a^*(\mathbf{y}^*) + \mathbf{I}) \mathrm{d}\,\Omega \qquad \text{(S.D2.45)}$$

References

Auriault, J.-L., Boutin, C., and Geindreau., C.: Homogenization of coupled phenomena in heterogenous media, Wiley-ISTE, London, 2009.