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The role of subtemperate slip in thermally driven ice stream margin migration

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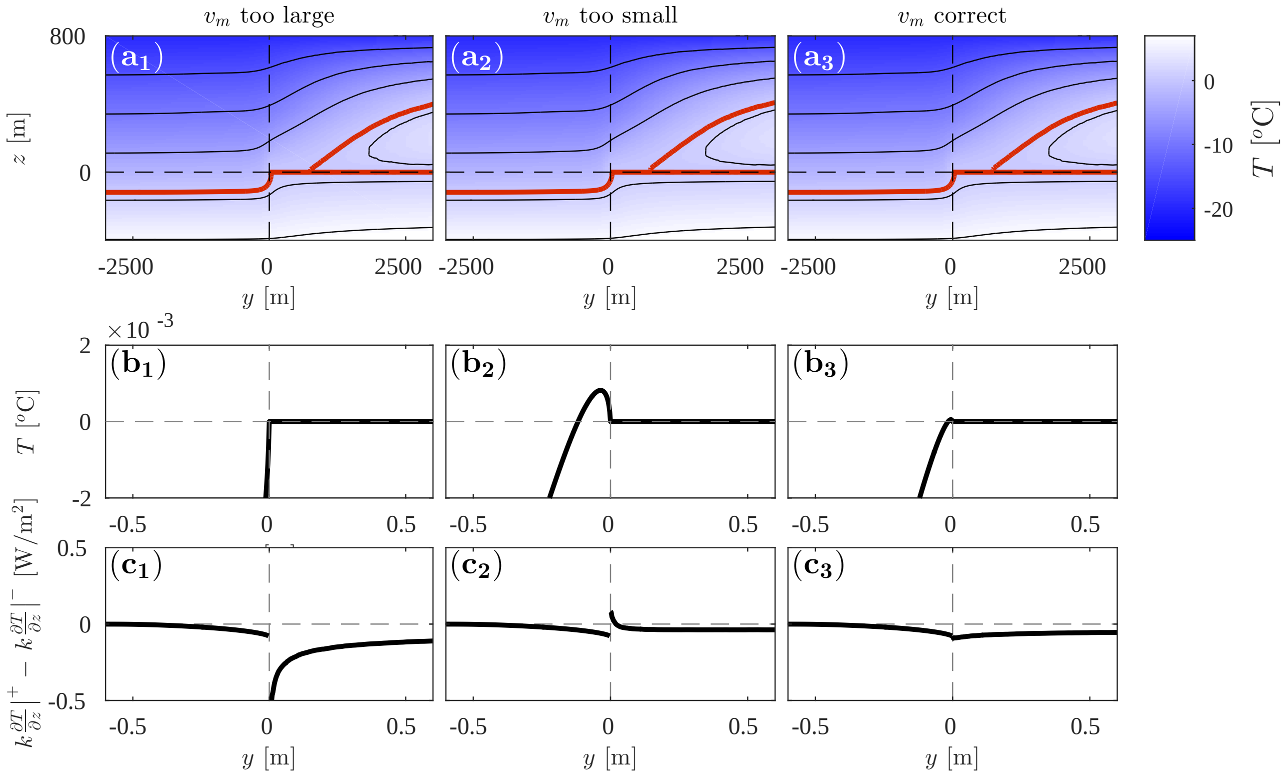

Figure S1: Numerical scheme to determine v_m. Each column (1-3) shows the temperature field (row a), temperature at the bed (row b) and net heat flux $k(\partial T/\partial z|^+ - \partial T/\partial z|^-)$ into the bed (row c). Note that $k(\partial T/\partial z|^+ - \partial T/\partial z|^-) = -\tau_c\sqrt{u^2+v^2}$ for $y < 0$. Temperature contours are plotted in 5^{o}C intervals, with $T = 0^o$C marked with a bold red line. Column 1 shows results for $v_m = 0.70$ m/year and an apparently singular heat flux at the origin in panel c$_1$. Column 2 shows results for $v_m = 0.63$ m/year with constraint (17a)$_1$ violated in panel b$_2$. Column 3 shows results for $v_m = 0.65$ m/year, satisfying both constraints. Note that the results in rows b and c are plotted for a narrow range of y around the origin. The region in which the inequality constraints are violated can be quite small even for substantially incorrect values of v_m. This underlines the need for a high grid resolution around the origin in our computations. Calculations were done with $h_s = 800$ m, $q_r = 10^4$ m year^{-1}, $T_s = -20^c$C, $A = 10^{-16}$, $q_{\text{geo}} = 50$ mW m^{-2}, $\tau_s = 200$ kPa, and $\tau_c = 5\tau_s = 1000$ kPa.

S1 Numerical scheme to determine the migration velocity

To illustrate how the migration rate is calculated, we show in figure S1 solutions to the heat flow problem without the inequality constraints (17a)–(17b) imposed. Figure S1 deliberately focuses on flow with subtemperate slip; for the case of a no-slip-to-free-slip transition, see figure 4.3 of Haseloff (2015) and figures 2 and 5 of Schoof (2012).

For ease of interpretation, we assume here (and in all other plots of the temperature field) that the melting point is at $T_m = 0^o$C. The three columns in the figure correspond to different migration rates $v_m = \partial y_m/\partial t$. For each migration rate, the first row of panels (a$_1$–a$_3$) shows the resulting temperature field in the ice, the second row (b$_1$–b$_3$) shows the temperature $T(y, 0)$ at the bed, and the third (c$_1$–c$_3$) shows $k(\partial T/\partial z|^+ - \partial T/\partial z|^-)$. On the cold side of the bed ($y < 0$), this equals $-\tau_c\sqrt{u^2+v^2}$. On the warm side of the bed, we require $k(\partial T/\partial z|^+ - \partial T/\partial z|^-)$ to be finite.

In column 1, we show a calculation in which v_m is set to a value that is too large. This leads to a negative singular rate of melting for $y > 0$ (panel c$_1$), or in other words, a singular rate of freezing. By contrast, the middle (column 2) shows a case where v_m is too small. This results in temperatures exceeding the melting point in a small region on the supposedly frozen side of the transition ($y < 0$, see panel b$_2$). In column 3, we show results with a value of v_m for which the temperature is below the freezing point for $y < 0$ and the freezing rate remains non-singular close to the origin. As discussed in appendix A and in the next section S2, this is the best we can hope for if we allow for slip with a finite amount of basal friction τ_c on the cold side of the transition: it is then not possible to suppress freezing completely.

In the present case, we cannot prove mathematically that there is a single migration rate for which neither inequality constraint in (17a)–(17b) is violated. Such a proof was however possible in the simpler version of our model in Schoof (2012), and computationally we find a unique v_m within bounds that are controlled by grid resolution. In practice, we determine the migration speed v_m iteratively using an adapted bisection method. The upper limit of the search interval is a migration velocity that is too large and therefore leads to a singular freezing rate on the ice stream side for $y > 0$ which violates $(17b)_2$ (as in column 1 of figure S1). The lower limit of the search interval has temperatures at or above the melting point for $y < 0$, violating $(17a)_1$ (as in column 2 of figure S1). As with a standard bisection method, we halve the search interval at every iteration. We determine in which interval to continue the search based on which inequality constraint is violated at the midpoint: if $(17a)_1$ is violated we continue in the upper half, otherwise in the lower half (see also Haseloff et al., 2015).

S2 Velocity, shear heating and temperature close to the cold-temperate transition

Here we extend the analysis of shear heating and temperature fields in appendix A of Schoof (2012) to the case of a transition from slip at a fixed basal yield stress τ_c to free stress. Our purpose is to demonstrate mathematically that the temperature field near the origin (assumed to be the location at which the cold-temperate transition takes place) allows only the three different cases described above:

1) positive temperatures for $y < 0$, conflicting with the assumption that the bed there is subtemperate, and subtemperate sliding is taking place

2) an infinite heat flux out of the bed, corresponding to an infinite rate of basal freezing on the warm side of the origin $y > 0$

3) as a limiting case, a finite rate of freezing on the warm side of the bed, equal to the dissipation rate on the subtemperate side of the bed

The numerical scheme in the previous section S1 is built on the assumption that the limiting case 3 is the only physically acceptable one.

For simplicity, we restrict ourselves to the case of constant ice viscosity η, and consider only flow parallel to the margin, assuming that the velocity component in the direction is much larger than the transverse velocity and therefore dominates the shear heating rate. We can treat the velocity as being the sum of a constant sliding velocity $\bar{u}_b$ at the transition from frictional to free slip, and a correction $\widetilde{u}(y, z)$. The latter then satisfies the Stokes flow problem

$$\eta \nabla^2 \widetilde{u} = 0$$

for $z > 0$, where ∇ is the gradient operator in the transverse y–z-plane, with boundary conditions

$$\eta \frac{\partial \widetilde{u}}{\partial z} = \begin{cases} \tau_c & \text{at } z = 0,\ y < 0 \\ 0 & \text{at } z = 0,\ y > 0. \end{cases}$$

A general solution can be derived using complex variables, letting $\zeta = y + iz$, and using the differentiation rules (England, 1971)

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial \zeta} + \frac{\partial}{\partial \bar{\zeta}} \qquad \frac{\partial}{\partial z} = i\left(\frac{\partial}{\partial \zeta} - \frac{\partial}{\partial \bar{\zeta}}\right). \tag{S1}$$

Since $\widetilde{u}$ satisfies Laplace's equation, it is the real part of a holomorphic function $\phi(\zeta)$, $\widetilde{u}(y, z) = \mathrm{Re}(\phi(\zeta))$, and we have $\partial\widetilde{u}/\partial y + i\partial\widetilde{u}/\partial z = \phi'(\zeta)$ (England, 1971). Continuing ϕ' analytically to the lower half-plane $\Im(\zeta) < 0$ by defining $\phi'(\zeta) = \overline{\phi'(\bar{\zeta})}$ (note that ϕ' has no physical meaning in the lower half-plane), we find that the extended function ϕ' is analytic in the ζ-plane cut along the negative half of the real axis, where it satisfies $i(\phi'^{+}(y) - \phi'^{-}(y)) = 2\tau_c$. The superscripts $+$ and $-$ indicate limits taken from above and below, respectively. Hence an integrable solution takes the general form (Muskhelishvili, 1992)

$$\phi'(\zeta) = -\frac{\tau_c}{\pi\eta}\log(\zeta) + \sum_{n=0}^{\infty} c_n \zeta^n,$$

where log is the usual branch of the natural logarithm with a branch cut on the negative real axis, and the c_n must be real to ensure the requisite symmetry of ϕ'. The corresponding velocity field expressed in polar coordinates, with $y = r\cos(\vartheta)$ and $z = r\sin(\vartheta)$, is

$$\widetilde{u} = \frac{\tau_c}{\pi\eta}\left\{r\vartheta\sin(\vartheta) - r[\log(r) - 1]\cos(\vartheta)\right\} + \sum_{n=0}^{n} \frac{c_n}{n+1} r^{n+1}\cos[(n+1)\vartheta].$$

Next, we consider the heat transport problem. At short enough length scales, several simplifications can be made. To an error of $O(\mathrm{Per})$, advection can be omitted, and the strain heating rate $\eta|\nabla\widetilde{u}|^2$ can be approximated by retaining only the first two terms in the solution for $\phi' \sim -\tau_c/(\pi\eta)\log(\zeta) + c_0$. In computing frictional dissipation due to sliding at the bed, we can also approximate the sliding velocity by $\bar{u}_b$ to an error of $O(r\log(r))$. Hence, to an error of that magnitude,

$$-k\nabla^2 T = \begin{cases} \frac{\tau_c^2}{\pi^2\eta}\left[\log(r/r_0)^2 + \vartheta^2\right] & \text{for } z > 0, \\ 0 & \text{for } z < 0, \end{cases} \tag{S2}$$

with the boundary conditions

$$T(y, 0) = 0 \qquad \text{for } z = 0,\ y > 0, \tag{S3}$$

$$-k\left[\frac{\partial T}{\partial z}\right]_{-}^{+} = \tau_c \bar{u}_b \qquad \text{for } z = 0,\ y < 0, \tag{S4}$$

$$[T(y, 0)]_{-}^{+} = 0 \qquad \text{for } z = 0,\ y < 0 \tag{S5}$$

where $\log(r_0) = c_0\pi\eta/\tau_c$. Importantly, the heat production rate for the no-slip to free-slip transition in Schoof (2012) behaves as $1/r$, whereas it has only a logarithmic singularity in r here.

Using (S1), we can express Poisson's equation (S2) in terms of ζ as

$$-4k\frac{\partial^2 T}{\partial\zeta\partial\overline{\zeta}} = \begin{cases} \frac{\tau_c^2}{\pi^2\eta}\log(\zeta/r_0)\log(\overline{\zeta}/r_0) & \text{for } \Im(\zeta) > 0 \\ 0 & \text{for } \Im(\zeta) < 0. \end{cases} \tag{S6}$$

We can write the solution in the form

$$\begin{aligned} T = &-\frac{\tau_c^2}{8\pi^2 k\eta}\Big\{2\left[\zeta\log(\zeta/r_0) - \zeta\right]\left[\overline{\zeta}\log(\overline{\zeta}/r_0) - \overline{\zeta}\right] \\ &- \left[\zeta\log(\zeta/r_0) - \zeta\right]^2 - \left[\overline{\zeta}\log(\overline{\zeta}/r_0) - \overline{\zeta}\right]^2 \\ &+ 2i\pi\left[\zeta^2\log(\zeta/r_0) - \overline{\zeta}^2\log(\overline{\zeta}/r_0) + \overline{\zeta}^2 - \zeta^2\right] + i\pi\left(\zeta^2 - \overline{\zeta}^2\right)\Big\} \\ &+ i\frac{\tau_c\bar{u}_b}{2k}\left(\zeta - \overline{\zeta}\right) + \varphi(\zeta) + \overline{\varphi(\zeta)} && \text{for } \Im(\zeta) > 0 \\ T = &\varphi(\zeta) + \overline{\varphi(\zeta)} && \text{for } \Im(\zeta) < 0 \end{aligned}$$

where φ is an analytic function in the lower and upper half planes, its form to be determined by the boundary conditions at the bed, where $\Im(\zeta) = 0$. Along the negative half of the real axis, the boundary conditions (S4) and (S5) written in complex variable form using (S1) together ensure that φ' and therefore φ are continuous across that boundary and hence analytic on the ζ-plane cut along the positive real axis. On that branch cut, $\varphi^{+}(y) + \overline{\varphi^{+}(y)} = \varphi^{-}(y) + \overline{\varphi^{-}(y)} = 0$. Splitting φ into a symmetric and antisymmetric part as $\Omega(\zeta) = [\varphi(\zeta) + \overline{\varphi(\overline{\zeta})}]/2$ and $\Psi(\zeta) = [\varphi(\zeta) - \overline{\varphi(\overline{\zeta})}]/2$, it is then straightforward to show that Ψ is analytic in the entire ζ plane, while Ω satisfies the homogeneous Hilbert problem

$$\Omega^+(y) + \Omega^-(y) = 0$$

on the positive half of the real axis. Requiring an integrable heat flux φ', we have a general solution

$$\varphi(\zeta) = \Omega(\zeta) + \Psi(\zeta) = -\zeta^{1/2} \sum_{n=0}^{\infty} \frac{ia_n}{2} \zeta^n - \sum_{n=0}^{\infty} \frac{ib_n}{2} \zeta^n$$

where $\zeta^{1/2}$ has a branch cut on the positive half of the real axis, the limit taken from above being the usual positive square root $\sqrt{y}$, and the a_n and b_n are purely real to satisfy the symmetries of Ω and Ψ.

To an error of $O(r^{5/2})$, we therefore obtain a temperature field close to the origin of the form

$$\begin{aligned} T(r,\vartheta) =& \frac{\tau_c^2}{4\pi^2 k\eta} r^2 \Bigg\{ \left[\left(\log\left(\frac{r}{r_0}\right) - 1 \right)^2 + \vartheta^2 \right] - \cos(2\vartheta) \left[\left(\log\left(\frac{r}{r_0}\right) - 1 \right)^2 - \vartheta^2 \right] \\ & + 2(\vartheta - \pi)\sin(2\vartheta) \left(\log\left(\frac{r}{r_0}\right) - 1 \right) - \pi \sin(2\vartheta) - 2\pi\vartheta\cos(2\vartheta) \Bigg\} - \frac{\tau_c \bar{u}_b}{k} r \sin(\vartheta) \\ & + a_0 r^{1/2} \sin\left(\frac{\vartheta}{2}\right) + a_1 r^{3/2} \sin\left(\frac{3\vartheta}{2}\right) + b_1 r \sin(\vartheta) + b_2 r^2 \sin(2\vartheta) \end{aligned}$$

for $0 < \vartheta < \pi$, and

$$T(r,\vartheta) = a_0 r^{1/2} \sin\left(\frac{\vartheta}{2}\right) + a_1 r^{3/2} \sin\left(\frac{3\vartheta}{2}\right) + b_1 r \sin(\vartheta) + b_2 r^2 \sin(2\vartheta)$$

for $\pi < \vartheta < 2\pi$. The response to englacial shear heating is represented by the term in curly brackets, which behaves as $O(r^2 \log(r)^2)$. The temperature is therefore dominated by the terms in the solution to the problem without englacial heating, of the form

$$T \sim a_0 r^{1/2} \sin(\vartheta/2) + a_1 r^{3/2} \sin(3\vartheta/2) + b_1 r \sin(\vartheta) + b_2 r^2 \sin(2\vartheta) - \begin{cases} \frac{\tau_c \bar{u}_b}{k} r \sin(\vartheta) & \text{for } 0 < \vartheta < \pi \\ 0 & \text{otherwise.} \end{cases}$$

As in Schoof (2012), it is easy to see that we require $a_0 \leq 0$ to ensure temperatures do not go above freezing at the bed on the cold side of the transition point (that is, on $\vartheta = \pi$, where the leading order form of T is then $T \sim a_0 r^{1/2}$). A consequence of this is that, with $a_0 \neq 0$, we obtain a singular heat flux $-kr^{-1}\partial T/\partial\vartheta|_{\vartheta=2\pi}^{\vartheta=0} = k a_0 r^{-1/2}$ out of the bed on the warm side.

If we assume that a singular heat flux out of the bed is not viable as it leads to freezing of the bed on the warm side of the transition, contradicting the assumption that the ice stream is widening, then we must have $a_0 = 0$. The temperature field near the origin is then linear at leading order, and can be written as $T \sim b_1 z$ for $z < 0$, $T \sim [b_1 - \tau_c \bar{u}_b/k]z$ for $z > 0$, with the horizontal temperature gradient only appearing at the next (higher) order.

There are two important conclusions that can be drawn from this. The first is that the net heat flux out of the bed is

$$-\frac{1}{r}\left.\frac{\partial T}{\partial \vartheta}\right|_{2\pi}^{0} = -\frac{1}{r}\left.\frac{\partial T}{\partial \vartheta}\right|_{\pi^+}^{\pi^-} = \tau_c \bar{u}$$

on both, the cold and the warm sides of the transition: it is impossible for the temperature gradient to change discontinuously from the left to the right of the transition point. The fact that the frictional heat $\tau_c \bar{u}_b$ generated to the left of the transition must be removed from the bed means that heat is removed at the same rate from the right, where presumably it must be supplied in the form of latent heat transported by drainage of meltwater along the bed. The second observation is that it is no longer necessary to have a region of temperate ice form near the transition point: if the temperature below the bed is above the melting point, we expect $b_1 < 0$ and hence $\partial T/\partial z < 0$ everywhere above the bed, corresponding to temperatures below the melting point in the ice.

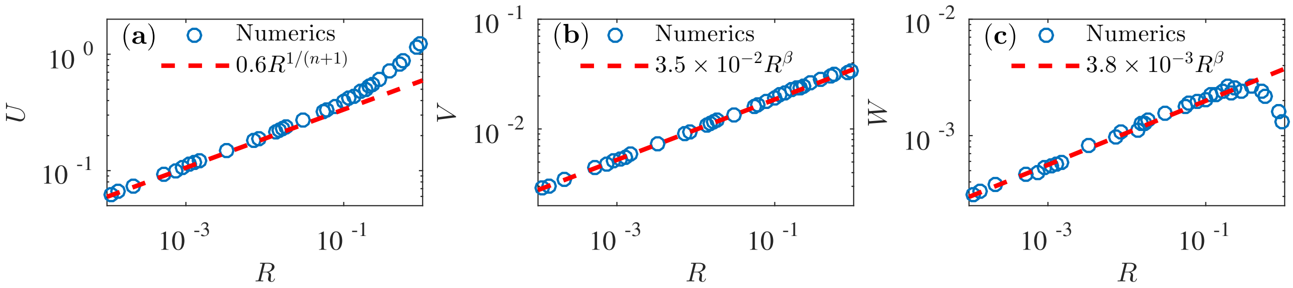

Figure S2: Comparison of numerical velocity solutions with asymptotic solutions from Rice (1967) and the solutions of the boundary value problem (S19)–(S21) for $\vartheta = \pi/8$. Panel a shows solutions of the downstream velocity U, panel b shows solutions of the across-stream velocity V and panel c shows solutions of the vertical velocity W. $n = 3$ in all three cases.

S3 The velocity field close to the transition from no slip to free slip

In section 4.3 we analyze the behavior of the temperature field close to the transition from no slip to free slip. To do so, we need to know the behavior of the velocities close to the origin, which we consider here. Near the origin of our geometry, i.e. for $R = (Y^2 + Z^2)^{1/2} \to 0$ and for $\varepsilon \ll 1$, the equation for the down-stream velocity (22) with boundary conditions (29)–(30a) is identical to the model for a crack-tip considered in Rice (1967, 1968). He shows that in polar coordinates, the velocity solution close to the transition from free slip to no slip is of the form

$$U \sim C_u R^{\frac{1}{n+1}} \sqrt{\frac{2n}{n+1} A_\vartheta^{\frac{2}{n+1}} + \cos\vartheta A_\vartheta^{\frac{1-n}{1+n}}} \qquad \text{for } R \to 0, \tag{S7}$$

where $R = \sqrt{Y^2 + Z^2}$, $\cos\vartheta = Y/R$, C_u a constant that depends on the far field conditions, and

$$A_\vartheta = \frac{n^2-1}{4n}\cos\vartheta + \sqrt{\left(\frac{n^2-1}{4n}\right)^2 \cos^2\vartheta + \frac{(n+1)^2}{4n}}. \tag{S8}$$

Figures S2a and S3a confirm that our numerical solution reproduces this behavior as $R \to 0$. From (S7) the asymptotic behavior of the heat production rate (32) is

$$\mathcal{A} \sim \left(\frac{C_u}{2}\right)^{1+1/n} R^{-1} A_\vartheta^{-1}. \tag{S9}$$

The important feature of this result is that the heat production is singular, behaving as R^{-1} near the transition point. This is not a surprise: a similar behavior for $n = 1$ appears in Schoof (2004, 2012) and for $n = 3$ in Suckale et al. (2014). For the frequently used special cases of $n = 1$ and $n = 3$, $\mathcal{A}$ can alternatively be written as

$$\mathcal{A} \sim C_a R^{-1} \times \begin{cases} \text{const.} & \text{for } n = 1, \\ \left(\sqrt{3 + \cos^2\vartheta} + \cos\vartheta\right)^{-1} & \text{for } n = 3. \end{cases} \tag{S10}$$

The local behavior of the across-stream velocities (V, W) is more difficult to determine. For a constant viscosity ($n = 1$), Barcilon and MacAyeal (1993) show that

$$V \sim C_v R^{1/2}\left(2\cos\frac{\vartheta}{2} + \sin\vartheta \sin\frac{\vartheta}{2}\right), \qquad W \sim -C_w R^{1/2} \sin\frac{\vartheta}{2}\cos^2\frac{\vartheta}{2}. \tag{S11}$$

For $n \neq 1$, the problem of finding the local behavior of V and W is complicated by the fact that the viscosity is determined by $|\nabla U|$, where the local behavior of U is given by (S7). To find a generalization

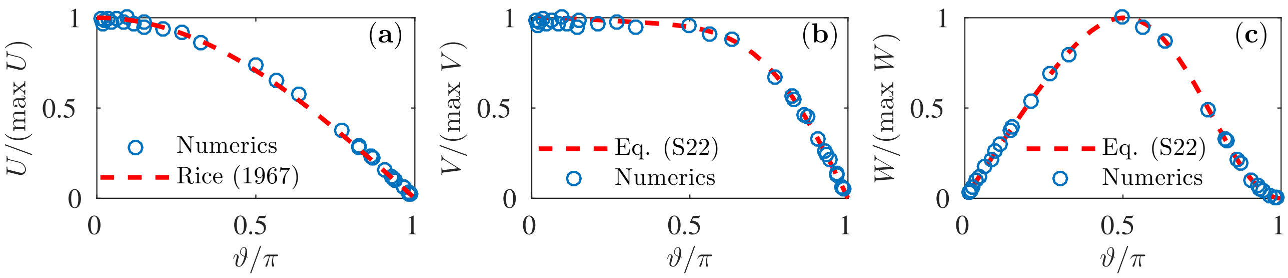

Figure S3: Comparison of numerical velocity solutions with asymptotic solutions from Rice (1967) and the solutions of the boundary value problem (S19)–(S21) for $R = 0.01$. Panel a shows scaled solutions of the downstream velocity U, panel b shows scaled solutions of the across-stream velocity V and panel c shows solutions of the vertical velocity W. $n = 3$ in all three cases.

of (S11) for $n \neq 1$, we rewrite (23) in polar coordinates (R, ϑ):

$$-\frac{\partial P}{\partial R} + \frac{1}{R}\frac{\partial}{\partial R}(R\Sigma_{RR}) + \frac{1}{R}\frac{\partial \Sigma_{\vartheta R}}{\partial \vartheta} - \frac{\Sigma_{\vartheta\vartheta}}{R} = 0, \tag{S12a}$$

$$-\frac{1}{R}\frac{\partial P}{\partial \vartheta} + \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\Sigma_{\vartheta R}\right) + \frac{1}{R}\frac{\partial \Sigma_{\vartheta\vartheta}}{\partial \vartheta} = 0, \tag{S12b}$$

$$\frac{1}{R}\frac{\partial}{\partial R}(RV_R) + \frac{1}{R}\frac{\partial V_\vartheta}{\partial \vartheta} = 0. \tag{S12c}$$

Here V_R and V_ϑ are the radial and angular velocity components, respectively, i.e., $\mathbf{V} = V_R\mathbf{e}_R + V_\vartheta\mathbf{e}_\vartheta$. The constitutive relations for the stresses Σ in polar coordinates are:

$$\Sigma_{RR} = \mu\frac{\partial V_R}{\partial R}, \qquad \Sigma_{\vartheta\vartheta} = \mu\frac{1}{R}\left(\frac{\partial V_\vartheta}{\partial \vartheta} + V_R\right), \qquad \Sigma_{\vartheta R} = \frac{1}{2}\mu\left(\frac{1}{R}\frac{\partial V_R}{\partial \vartheta} + \frac{\partial v_\vartheta}{\partial R} - \frac{v_\vartheta}{R}\right). \tag{S13}$$

The boundary conditions (29) and (30a) at the base become

$$V_\vartheta = \mu\frac{1}{R}\frac{\partial V_R}{\partial \vartheta} = 0 \quad \text{for } \vartheta = 0, \qquad V_\vartheta = V_R = 0 \quad \text{for } \vartheta = \pi. \tag{S14}$$

The downstream velocity U, given by (S7)–(S8) determines the viscosity μ through

$$\mu \sim R^{\frac{1-n}{1+n}}N \qquad \text{with} \qquad N(\vartheta) = [A_\vartheta(\vartheta)]^{\frac{n-1}{n+1}}. \tag{S15}$$

We put $\mu = R^{\frac{1-n}{1+n}}N$ and make the *ansatz* $(V_R, V_\vartheta) = R^\beta(\bar{V}_R(\vartheta), \bar{V}_\vartheta(\vartheta))$ and $P = R^{\beta-2/(n+1)}P_\vartheta(\vartheta)$, which gives in (S12c)

$$\bar{V}_R + \frac{1}{\beta+1}\bar{V}_\vartheta' = 0. \tag{S16}$$

Here a prime denotes an ordinary derivative with respect to ϑ, so $\bar{V}_\vartheta' = \mathrm{d}\bar{V}_\vartheta/\mathrm{d}\vartheta$. Equations (S12a)–(S12b) become

$$-a_0P_\vartheta - a_1N\bar{V}_\vartheta' + \left(a_2N\bar{V}_\vartheta - a_3N\bar{V}_\vartheta''\right)' = 0, \tag{S17a}$$

$$-P_\vartheta' + b_1N\bar{V}_\vartheta - b_2N\bar{V}_\vartheta'' + b_3\left(N'\bar{V}_\vartheta' + N\bar{V}_\vartheta''\right) = 0, \tag{S17b}$$

where

$$a_0 = \left[\beta - \frac{2}{n+1}\right], \qquad a_1 = \frac{\beta}{\beta+1}\left[\beta + \frac{2n}{n+1}\right], \qquad a_2 = \frac{(\beta-1)}{2}, \qquad a_3 = \frac{1}{2}\frac{1}{\beta+1},$$

$$b_1 = \frac{1}{2}\left(\beta + \frac{2n}{n+1}\right)(\beta-1), \qquad b_2 = \frac{1}{2}\left(\beta + \frac{2n}{n+1}\right)\frac{1}{\beta+1}, \qquad b_3 = \frac{\beta}{\beta+1}.$$

Elimination of the pressure in (S17a) by use of (S17b) leads to a fourth order homogeneous differential equation for $\bar{V}_\vartheta$ with non-constant coefficients

$$0 = \left(b_1 + c_5\frac{N''}{N}\right)\bar{V}_\vartheta + \left(c_4\frac{N'}{N} - c_2\right)V_\vartheta' + \left(c_3 - \frac{N''}{N}\right)\bar{V}_\vartheta'' - 2\frac{N'}{N}\bar{V}_\vartheta''' - \bar{V}_\vartheta'''' \tag{S19}$$

where

$$c_1 = \frac{a_0}{a_3}b_1, \quad c_2 = \frac{a_1}{a_3}, \quad c_3 = \frac{a_0}{a_3}(b_2 - b_3) + \frac{a_2}{a_3}, \quad c_4 = \left[2\frac{a_2}{a_3} - \frac{a_1}{a_3} - \frac{a_0}{a_3}b_3\right], \quad c_5 = \frac{a_2}{a_3}, \tag{S20}$$

and N is given by equation (S15). The boundary conditions (S14) are likewise homogeneous,

$$\bar{V}_\vartheta = \bar{V}_\vartheta'' = 0 \quad \text{for } \vartheta = 0, \qquad\qquad \bar{V}_\vartheta = \bar{V}_\vartheta' = 0 \quad \text{for } \vartheta = \pi, \tag{S21}$$

and we have a generalized eigenvalue problem in which the eigenvalue β is somewhat unconventionally hidden in the coefficients (S20). We solve this problem using a shooting method, which gives $\beta = 0.271...$ as the lowest positive eigenvalue for $n = 3$. Once again we find that our numerical solutions reproduce this behavior, see figure S2b-c. Note that β is greater than $1/(1+n)$, so that the viscosity is indeed dominated by gradients of the downstream velocity U. The shooting method also gives us V_ϑ, from which V_R can be calculated through equation (S16). The velocity components (V, W) in Cartesian coordinates can be calculated from $(\bar{V}_R, \bar{V}_\vartheta)$ through

$$V = R^\beta(\bar{V}_R\cos\vartheta - \bar{V}_\vartheta\sin\vartheta), \qquad W = R^\beta(\bar{V}_R\sin\vartheta + \bar{V}_\vartheta\cos\vartheta). \tag{S22}$$

The angular dependence of U, V and W is shown in figure S3.

Note that the local solution we have derived here stems from a problem (equations (22)–(30b) of the main text) that contains no free parameters when — as we have assumed here — τ is infinite. As a result, we are guaranteed that C_a, C_u, $\bar{V}_R$ and $\bar{V}_\vartheta$ are also parameter-free, as is implied in the main text.

S4 The outer temperature problem for strong heat production

In the main text, the velocity field derived in section S3 above is used to construct a local advection-diffusion problem for heat transport near the cold-temperate (and no-slip-to-slip) transition. That local model, equations (40) of the main text, is mathematically a boundary layer. It only depends on Λ and $\widetilde{V}_m$ as parameters, suggesting that $\widetilde{V}_m = \widetilde{f}(\Lambda)$, if the far-field conditions on the boundary layer only depend on Λ, too. These boundary conditions mathematically come out of asymptotic matching with an 'outer' problem that describes heat transport at a larger scale (Holmes, 2013). Here we verify that matching leads to far-field conditions that only depend on Λ as required.

The outer problem to the conductive boundary layer itself describes heat flow in a slender region along the bed. To identify leading order terms in this outer problem (confusingly, itself a boundary layer to the advection-dominated heat transport across the bulk of the ice thickness), we first need to understand the transverse velocity field near the bed. $V = 0$ implies that $\partial V/\partial Y = 0$ at the bed, so $\partial W/\partial Z = 0$ by mass conservation. By Taylor expansion, we obtain $V \sim Z$, $W \sim Z^2$. Near-bed advection in the outer problem is captured by considering a thin region of vertical extent $Z_{\mathrm{Pe}} = \mathrm{Pe}^{-\beta/(1+\beta)} \ll 1$ relative to ice thickness, labeled the 'advective boundary layer' in figure 5. Within this region, we rescale $Z = Z_{\mathrm{Pe}}\widehat{Z}$, $V = Z_{\mathrm{Pe}}\widehat{V}$, $W = Z_{\mathrm{Pe}}^2\widehat{W}$, $\mathcal{A} = \widehat{\mathcal{A}}$, $\Theta = \widehat{\Theta}$.

Note that the vertical coordinate in the advective region is related to the vertical coordinate in the conductive boundary layer through $\widetilde{Z} = \Lambda^{-1}\mathrm{Pe}^{(1-\beta)/(1+\beta)}\widehat{Z}$. For $\beta < 1$, $\widehat{Z} = O(1)$ implies that $\widetilde{Z} \gg 1$. For $n = 1$, the exponent β equals $1/2$, and for $n = 3$ we have $\beta \approx 0.27$ (see supplementary section S3). Therefore the near-bed advective layer is a viable outer region to the conductive boundary layer because the advective layer has a much larger vertical and horizontal extent than the conductive boundary layer.

For $n = 3$ (or generally for $n > 1$ and $\beta < 1/2$), the outer problem is,

$$\widetilde{V}_m \frac{\partial \widehat{\Theta}}{\partial Y} + \Lambda \left(\widehat{V} \frac{\partial \widehat{\Theta}}{\partial Y} + \widehat{W} \frac{\partial \widehat{\Theta}}{\partial \widehat{Z}} \right) = a \quad \text{for} \quad 0 < \widehat{Z}, \tag{S23a}$$

$$\widetilde{V}_m \frac{\partial \widehat{\Theta}}{\partial Y} = 0 \quad \text{for} \quad \widehat{Z} < 0, \tag{S23b}$$

to an error of $O(\mathrm{Pe}^{(2\beta-1)/(1+\beta)})$. As required, (S23a)–(S23b) only depend on $\widetilde{V}_m$ and Λ. As we are considering an outer problem that describes a slender region near the bed, our choice of reduced temperature Θ means that the relevant boundary condition is $\widehat{\Theta}(\widehat{Z} = 0) \to 0$ as $Y \to -\infty$, equation (34c), which equally does not depend on any additional parameters.

S5 Mechanical problem for a small slip region: $\tau \sim \alpha^{1/(n+1)} \gg 1$

When we allow for subtemperate sliding, but at a large basal yield stress $\tau \gg 1$, the velocity field will change only by a small amount: over most of the domain, basal shear stress will not attain the yield stress. The only location where that is not the case is close to the origin, where a hard transition from slip to no slip would lead to a stress singularity, exceeding any finite yield stress. In other words, the region of slip created by a large but finite τ is a mechanical boundary layer close to the origin, which remains small compared with the ice thickness. Outside that boundary layer, the velocity field will remain unchanged. In fact, at length scales that are intermediate between the boundary layer and the ice thickness scales, the local solution of supplementary section S4 will still apply, and provides the appropriate matching conditions on the mechanical boundary layer created by the small slip region. In this section, we construct a leading order model for that boundary layer. We focus on the case of $\tau \sim \alpha^{1/(n+1)} \gg 1$, in which the size of this mechanical boundary layer is the same as the size of the thermal boundary layer: this is the minimum size of the mechanical boundary layer at which we expect to start seeing an effect of subtemperate sliding on margin migration.

We rescale the mechanical field equations using $(Y, Z) = R_\alpha(\widetilde{Y}, \widetilde{Z})$, $\mathcal{A} = R_\alpha{}^{-1}\widetilde{\mathcal{A}}$, $U = R_\alpha^{1/(n+1)}\widetilde{U}$, $(V, W) = R_\alpha{}^{\beta}(\widetilde{V}, \widetilde{W})$, and $P = R_\alpha{}^{-1/(n+1)}\widetilde{P}$ where $R_\alpha = \alpha^{-1}$. The choice of exponent β ensures that the boundary layer solution can be matched with the outer problem at the ice thickness scale, whose behavior in the matching region (Holmes, 2013) is given by supplementary section S4 as discussed. This yields an equation for the velocity in the downstream direction of the same form as (22):

$$\frac{\partial}{\partial \widetilde{Y}} \left(\widetilde{\mu} \frac{\partial \widetilde{U}}{\partial \widetilde{Y}} \right) + \frac{\partial}{\partial \widetilde{Z}} \left(\widetilde{\mu} \frac{\partial \widetilde{U}}{\partial \widetilde{Z}} \right) = 0. \tag{S24}$$

In the across-stream direction, we obtain from (23)

$$\frac{\partial}{\partial \widetilde{Y}} \left(2\widetilde{\mu} \frac{\partial \widetilde{V}}{\partial \widetilde{Y}} \right) + \frac{\partial}{\partial \widetilde{Z}} \left[\widetilde{\mu} \left(\frac{\partial \widetilde{V}}{\partial \widetilde{Z}} + \frac{\partial \widetilde{W}}{\partial \widetilde{Y}} \right) \right] - \frac{\partial \widetilde{P}}{\partial \widetilde{Y}} = 0, \tag{S25a}$$

$$\frac{\partial}{\partial \widetilde{Y}} \left[\widetilde{\mu} \left(\frac{\partial \widetilde{V}}{\partial \widetilde{Z}} + \frac{\partial \widetilde{W}}{\partial \widetilde{Y}} \right) \right] + \frac{\partial}{\partial \widetilde{Z}} \left(2\widetilde{\mu} \frac{\partial \widetilde{W}}{\partial \widetilde{Z}} \right) - \frac{\partial \widetilde{P}}{\partial \widetilde{Z}} = 0, \tag{S25b}$$

$$\frac{\partial \widetilde{V}}{\partial \widetilde{Y}} + \frac{\partial \widetilde{W}}{\partial \widetilde{Z}} = 0. \tag{S25c}$$

μ is the rescaled non-dimensional viscosity

$$\widetilde{\mu} = \frac{1}{2^{1/n}} \left[\left| \frac{\partial \widetilde{U}}{\partial \widetilde{Y}} \right|^2 + \left| \frac{\partial \widetilde{U}}{\partial \widetilde{Z}} \right|^2 \right]^{\frac{1-n}{2n}}. \tag{S26}$$

As before, we find for the vertical velocity component along the bed

$$\widetilde{W} = 0 \quad \text{at} \quad \widetilde{Z} = 0. \tag{S27}$$

Similarly, the free slip boundary condition (29) on the temperate side remains unchanged

$$\widetilde{\mu}\frac{\partial \widetilde{U}}{\partial \widetilde{Z}} = \widetilde{\mu}\frac{\partial \widetilde{V}}{\partial \widetilde{Z}} = 0 \quad \text{at } \widetilde{Z} = 0, \quad \widetilde{Y} > 0. \tag{S28}$$

On the frozen side of the bed, we have from (30b)

$$\left.\begin{array}{rlll} \text{either} & \widetilde{\mu}\frac{\partial \widetilde{U}}{\partial \widetilde{Z}} = \alpha^{1/(n+1)}\tau\frac{\widetilde{U}}{|\widetilde{U}|}, & \widetilde{\mu}\frac{\partial \widetilde{V}}{\partial \widetilde{Z}} = \alpha^{1/(n+1)}\tau\frac{\widetilde{V}}{|\widetilde{U}|}, & |\widetilde{U}| > 0, \quad |\widetilde{V}| > 0 \\ \text{or} & \left|\widetilde{\mu}\frac{\partial \widetilde{U}}{\partial \widetilde{Z}}\right| < \alpha^{1/(n+1)}\tau, & \left|\widetilde{\mu}\frac{\partial \widetilde{V}}{\partial \widetilde{Z}}\right| < \alpha^{1/(n+1)}\tau\left|\frac{\widetilde{V}}{\widetilde{U}}\right|, & |\widetilde{U}| = |\widetilde{V}| = 0 \end{array}\right\} \text{for } \widetilde{Y} < 0,\ \widetilde{Z} = 0. \tag{S29}$$

Equations (S24)–(S29) only depend on $\alpha^{1/(n+1)}\tau = \Gamma^{-(n+1)}$, as required for (46) to hold.

S6 Limit of large slip region: $\tau_c \ll \tau_s$

We conclude by considering the opposite parametric limit in τ to that considered above: we derive an otherwise elusive closed-form expression for V_m in the limit $\tau \ll 1$. When considering the case of small basal yield stress τ, the region of subtemperate slip becomes wide compared with ice thickness. Simultaneously, we consider the case of $\alpha \gg 1$, $\mathrm{Pe} \gg 1$, identifying the relevant distinguished limit as $\tau\mathrm{Pe} \sim \alpha^2 \gg 1$ later.

There are two rescalings required: first, for the mechanical problem and second, for the thermal problem. For the mechanical problem, we put

$$\widehat{Y} = \tau Y, \qquad \widehat{Z} = Z, \qquad \widehat{U} = \tau U, \qquad \widehat{V} = V, \qquad \widehat{W} = \tau^{-1}W, \qquad \widehat{P} = \tau^{-1}P. \tag{S30}$$

Under this rescaling, the mechanical problem in the boundary layer becomes

$$\tau^2\frac{\partial}{\partial \widehat{Y}}\left(\widehat{\mu}\frac{\partial \widehat{U}}{\partial \widehat{Y}}\right) + \frac{\partial}{\partial \widehat{Z}}\left(\widehat{\mu}\frac{\partial \widehat{U}}{\partial \widehat{Z}}\right) = 0, \tag{S31a}$$

$$\tau^2\frac{\partial}{\partial \widehat{Y}}\left(2\widehat{\mu}\frac{\partial \widehat{V}}{\partial \widehat{Y}}\right) + \frac{\partial}{\partial \widehat{Z}}\left[\widehat{\mu}\left(\frac{\partial \widehat{V}}{\partial \widehat{Z}} + \tau^2\frac{\partial \widehat{W}}{\partial \widehat{Y}}\right)\right] - \tau^2\frac{\partial \widehat{P}}{\partial \widehat{Y}} = 0, \tag{S31b}$$

$$\frac{\partial}{\partial \widehat{Y}}\left[\widehat{\mu}\left(\frac{\partial \widehat{V}}{\partial \widehat{Z}} + \tau^2\frac{\partial \widehat{W}}{\partial \widehat{Y}}\right)\right] + \frac{\partial}{\partial \widehat{Z}}\left(2\widehat{\mu}\frac{\partial \widehat{W}}{\partial \widehat{Z}}\right) - \frac{\partial \widehat{P}}{\partial \widehat{Z}} = 0, \tag{S31c}$$

$$\frac{\partial \widehat{V}}{\partial \widehat{Y}} + \frac{\partial \widehat{W}}{\partial \widehat{Z}} = 0, \tag{S31d}$$

where

$$\widehat{\mu} = \frac{1}{2^{1/n}}\left[\left(\frac{\partial \widehat{U}}{\partial \widehat{Y}}\right)^2 + \tau^{-2}\left(\frac{\partial \widehat{U}}{\partial \widehat{Z}}\right)^2\right]^{(1-n)/(2n)} \tag{S32}$$

for $0 < \widehat{Z} < 1$. Assume that there is slip for $\widehat{Y}_0 < \widehat{Y} < 0$, meaning $\widehat{U} > 0$ at $\widehat{Z} = 0$. In that region, we then have the following boundary conditions

$$\widehat{\mu}\frac{\partial \widehat{U}}{\partial \widehat{Z}} = 0, \qquad \widehat{\mu}\left(\frac{\partial \widehat{V}}{\partial \widehat{Z}} + \tau^2\frac{\partial \widehat{W}}{\partial \widehat{Y}}\right) = 0, \qquad \widehat{W} = 0 \qquad \text{for } \widehat{Y} > 0,\ \widehat{Z} = 0, \tag{S33a}$$

$$\widehat{\mu}\frac{\partial \widehat{U}}{\partial \widehat{Z}} = \tau^2, \qquad \widehat{\mu}\left(\frac{\partial \widehat{V}}{\partial \widehat{Z}} + \tau^2\frac{\partial \widehat{W}}{\partial \widehat{Y}}\right) = \tau^2\frac{\widehat{V}}{\widehat{U}}, \qquad \widehat{W} = 0 \qquad \text{for } \widehat{Y}_0 < \widehat{Y} < 0,\ \widehat{Z} = 0. \tag{S33b}$$

Expanding as $\widehat{U} = \widehat{U}^{(0)} + \tau^2\widehat{U}^{(1)} + \ldots$, $\widehat{V} = \widehat{V}^{(0)} + \tau^2\widehat{V}^{(1)} + \ldots$, $\widehat{W} = \widehat{W}^{(0)} + \tau^2\widehat{W}^{(1)} + \ldots$, we find that $\widehat{U}^{(0)} = \widehat{U}^{(0)}(\widehat{Y})$, $\widehat{V}^{(0)} = \text{constant}$, $\widehat{W}^{(0)} = 0$. In other words, a wide region of subtemperate slip implies that the plug flow of the ice stream extends past the thermal margin of the ice stream into a rapidly sliding but cold-based region. The axial velocity $\widehat{U}^{(0)}$ here satisfies the ice-stream-like model for a laterally sheared plug flow with constant basal drag:

$$\frac{\partial}{\partial \widehat{Y}}\left(\frac{1}{2^{1/n}}\left|\frac{\partial \widehat{U}^{(0)}}{\partial \widehat{Y}}\right|^{(1-n)/n}\frac{\partial \widehat{U}^{(0)}}{\partial \widehat{Y}}\right)-1=0$$

for the region $\widehat{Y}_0<\widehat{Y}<0$ where $\widehat{U}^{(0)}>0$ (this can be shown by vertical integration of (S31a), bearing in mind that $\widehat{\mu}\partial\widehat{U}/\partial\widehat{Z}=0$ at the ice stream surface at $\widehat{Z}=1$, (27)). One the ice stream side $\widehat{Y}>0$, we have no basal drag and so the equivalent model is

$$\frac{\partial}{\partial \widehat{Y}}\left(\frac{1}{2^{1/n}}\left|\frac{\partial \widehat{U}^{(0)}}{\partial \widehat{Y}}\right|^{(1-n)/n}\frac{\partial \widehat{U}^{(0)}}{\partial \widehat{Y}}\right)=0.$$

The original matching conditions with the ice stream as $Y\to\infty$ $(25)_1$ can then simply be reduced to a stress condition at $\widehat{Y}=0$,

$$\frac{1}{2^{1/n}}\left|\frac{\partial \widehat{U}^{(0)}}{\partial \widehat{Y}}\right|^{(1-n)/n}\frac{\partial \widehat{U}^{(0)}}{\partial \widehat{Y}}=1 \qquad \text{at } \widehat{Y}=0.$$

From (S31b) with $(\text{S33a})_2/(\text{S33b})_2$, we can see that the across-stream velocity $\widehat{V}^{(0)}$ has no vertical profile, either. Vertically integrating the mass balance equation (S31d) with $(\text{S33a})_3$ and $(\text{S33b})_3$ and (27), we can further show $\partial\widehat{V}^{(0)}/\partial\widehat{Y}=0$, or $\widehat{V}^{(0)}=\text{constant}$.

Matching with the region $\widehat{Y}<\widehat{Y}_0$, where there is no sliding, in principle requires a boundary layer around $\widehat{Y}<\widehat{Y}_0$ whose extent is comparable with ice thickness. The appropriate rescaling in that boundary layer is

$$\breve{Y}=Y-\tau^{-1}\widehat{Y}_0, \qquad \breve{Z}=Z, \qquad \breve{U}=\tau^{-1}U, \qquad \breve{V}=V, \qquad \breve{W}=W, \qquad \breve{P}=P. \tag{S34}$$

We do not give full detail of that boundary layer; the result of matching with (S31) and the far field as $\breve{Y}\to-\infty$ is simply the intuitive result that

$$\widehat{U}^{(0)}=\frac{\partial \widehat{U}^{(0)}}{\partial \widehat{Y}}=0, \qquad \widehat{V}^{(0)}=\int_0^1 1-(1-\widehat{Z})^{n+1}\,\mathrm{d}\widehat{Z}=\frac{n+1}{n+2} \qquad \text{at } \widehat{Y}=\widehat{Y}_0,$$

and we have a solution for the sliding velocity of the form

$$\widehat{U}^{(0)}=\frac{2(\widehat{Y}-\widehat{Y}_0)^{n+1}}{n+1},$$

with

$$\widehat{Y}_0=-1.$$

Putting $\widehat{T}=\mathcal{T}$, the corresponding thermal problem in the region with subtemperate slip is then at leading order in τ^2

$$\tau V_m\frac{\partial \widehat{T}}{\partial \widehat{Y}}+\mathrm{Pe}\tau\widehat{V}^{(0)}\frac{\partial \widehat{T}}{\partial \widehat{Y}}-\frac{\partial^2\widehat{T}}{\partial \widehat{Z}^2}=\frac{\alpha}{2^{1+1/n}}\left|\frac{\partial \widehat{U}}{\partial \widehat{Y}}\right|^{n+1} \qquad \text{for } 0<\widehat{Z}<1, \tag{S35a}$$

$$\gamma\tau V_m\frac{\partial \widehat{T}}{\partial \widehat{Y}}-\kappa\frac{\partial^2\widehat{T}}{\partial \widehat{Z}^2}=0 \qquad \text{for } \widehat{Z}<0 \tag{S35b}$$

subject to the jump conditions

$$\left[\widehat{T}\right]_-^+=0, \qquad -\left.\frac{\partial \widehat{T}}{\partial \widehat{Z}}\right|^+ +\kappa\left.\frac{\partial \widehat{T}}{\partial \widehat{Z}}\right|^- =\alpha\widehat{U}^{(0)} \qquad \text{at } \widehat{Z}=0, \quad \widehat{Y}_0<\widehat{Y}<0. \tag{S35c}$$

As before, we assume that $\alpha \gg 1$ and $\mathrm{Pe} \gg 1$. With $\alpha \gg 1$, we require a short vertical length scale α^{-1} to be able to conduct heat generated at the bed through frictional sliding into the ice, and a commensurately large migration velocity to balance vertical conduction at that scale. If we assume that lateral inflow can also contribute to energy balance at the same scale, we require the distinguished limit

$$\mathrm{Pe}\tau \sim \alpha^2$$

and can rescale as

$$\check{V}_m = \mathrm{Pe}^{-1} V_m, \qquad \check{Y} = \widehat{Y} - \widehat{Y}_0, \qquad \check{Z} = \alpha \widehat{Z}, \qquad \check{T} = \widehat{T} \tag{S36}$$

leading to the leading order diffusive boundary layer problem

$$\frac{\mathrm{Pe}\tau}{\alpha^2}\left(\widehat{V}^{(0)} + \check{V}_m\right)\frac{\partial \check{T}}{\partial \check{Y}} - \frac{\partial^2 \check{T}}{\partial \check{Z}^2} = 0 \qquad \text{for } 0 < \widehat{Z} < 1, \tag{S37a}$$

$$\gamma \frac{\mathrm{Pe}\tau}{\alpha^2}\check{V}_m \frac{\partial \check{T}}{\partial \check{Y}} - \kappa \frac{\partial^2 \check{T}}{\partial \check{Z}^2} = 0 \qquad \text{for } \widehat{Z} < 0, \tag{S37b}$$

subject to the jump conditions

$$\left[\check{T}\right]_-^+ = 0, \qquad -\left.\frac{\partial \check{T}}{\partial \check{Z}}\right|^+ + \kappa \left.\frac{\partial \check{T}}{\partial \check{Z}}\right|^- = \widehat{U}^{(0)} = \frac{2\check{Y}^{n+1}}{n+1} \qquad \text{at } \check{Z} = 0. \tag{S37c}$$

The outer problem in $\check{Z} = \alpha \widehat{Z}$ to this advection-diffusion boundary layer problem is simply the leading order (in $\alpha^2 \sim \mathrm{Pe}\tau$) version of (S35), which is the pure advection problem

$$\frac{\mathrm{Pe}\tau}{\alpha^2}\left(\widehat{V}^{(0)} + \check{V}_m\right)\frac{\partial \check{T}}{\partial \check{Y}} = 0 \qquad \text{for } 0 < \widehat{Z} < 1, \tag{S38a}$$

$$\gamma \frac{\mathrm{Pe}\tau}{\alpha^2}\check{V}_m \frac{\partial \check{T}}{\partial \check{Y}} = 0 \qquad \text{for } \widehat{Z} < 0, \tag{S38b}$$

leading to the conclusion that, outside the diffusive boundary layer with height above or below the bed described by $\check{Z} \sim O(1)$, we simply have the far-field temperature field advected from $\check{Y} = 0$.

From the rescaling above, we can immediately see that we expect

$$V_m = Pe \check{V}_m = \frac{\alpha^2}{\tau} f\left(\frac{\mathrm{Pe}\tau}{\alpha^2}, \gamma, \kappa\right)$$

for some function f (in fact, the dependence on κ and γ can be shown to collapse onto a dependence on the product $\kappa\gamma$ alone). It turns out we can compute the function f exactly, which we do below.

The boundary conditions (S37c) only hold up to $\check{Y} = -\widehat{Y}_0 = 1$. However, in the diffusion problem (S37), $\check{Y}$ is the time-like variable ($\check{Z}$ being space-like), and if we are only interested in the solution for $0 < \check{Y} < -\widehat{Y}_0$ (the region where subtemperate slip is possible), we can without loss of generality treat (S37) as applying for all $\check{Y} > 0$, which permits the problem to be solved by Laplace transforms. Define

$$\widetilde{f}(s) = \mathcal{L}(f)(s) = \int_0^\infty f(\check{Y}) \exp(-s\check{Y}) \, \mathrm{d}\check{Y}.$$

Then

$$\mathcal{L}\left(\check{Y}^{n+1}\right) = s^{-(n+2)}\Gamma(n+2)$$

where Γ is the standard gamma function. Let

$$\check{T} = \nu - 1 + \Theta,$$

so that (34c) becomes $\Theta = 0$ at $\check{Y} = 0$. Transforming (S37) gives

$$sv^{\pm}\widetilde{\Theta} - \frac{\partial^2 \widetilde{\Theta}}{\partial \check{Z}^2} = 0$$

with $v^+ = \mathrm{Pe}\tau(\breve{V}^{(0)} + \breve{V}_m)/\alpha^2$ for $\breve{Z} > 0$, $v^- = \gamma\mathrm{Pe}\tau\breve{V}_m/(\alpha^2\kappa)$ for $\breve{Z} < 0$, and

$$\left[\widetilde{\Theta}\right]_-^+ = 0, \qquad -\left.\frac{\partial\widetilde{\Theta}}{\partial\breve{Z}}\right|^+ - \kappa\left.\frac{\partial\widetilde{\Theta}}{\partial\breve{Z}}\right|^- = \frac{2s^{-(n+2)}\Gamma(n+2)}{n+1} \qquad \text{at } \breve{Z} = 0.$$

Matching the outer problem additionally requires $\widetilde{\Theta} \to 0$ as $\breve{Z} \to \pm\infty$. This has solution

$$\widetilde{\Theta} = A\exp\left(\mp\sqrt{sv^{\pm}}\breve{Z}\right),$$

the upper sign being chosen consistently for $\breve{Z} > 0$, the lower for $\breve{Z} < 0$. The flux condition at $\breve{Z} = 0$ requires that

$$A\left(\sqrt{v^+ s} + \kappa\sqrt{v^- s}\right) = \frac{2s^{-(n+2)}\Gamma(n+2)}{n+1}.$$

so that the Laplace transform of Θ at the bed is given by

$$\left.\widetilde{\Theta}\right|_{\breve{Z}=0} = A = \frac{2s^{-(n+5/2)}\Gamma(n+2)}{(n+1)\left(\sqrt{v^+} + \kappa\sqrt{v^-}\right)}.$$

We can now take the inverse Laplace transform; by inspection,

$$\Theta(\breve{Y}, 0) = \frac{2\Gamma(n+2)}{(n+1)\Gamma(n+5/2)\left(\sqrt{v^+} + \kappa\sqrt{v^-}\right)}\breve{Y}^{n+3/2}.$$

At $\breve{Y} = -\widehat{Y}_0 = 1$, we must have temperature reaching the melting point $\breve{T} = 0$, which becomes $\Theta = 1 - \nu$, so the migration velocity is determined by

$$\frac{2\Gamma(n+2)}{(n+1)\Gamma(n+5/2)\left(\sqrt{v^+} + \kappa\sqrt{v^-}\right)} = 1 - \nu,$$

or, using the definition of $v^{\pm}$,

$$\frac{2\Gamma(n+2)}{(n+1)\Gamma(n+5/2)}\frac{\alpha}{(1-\nu)\sqrt{\mathrm{Pe}\tau}} = \sqrt{\widehat{V}^{(0)} + \breve{V}_m} + \sqrt{\kappa\gamma\breve{V}_m}.$$

This is solvable in closed form; here we give only the (relatively simpler) solution for $\kappa\gamma = 1$, the case also considered in the main paper. Then, also recalling that $\breve{V}_m = \mathrm{Pe}^{-1}V_m$ and $\widehat{V}^{(0)} = (n+1)/(n+2)$, we can find the original migration velocity V_m as

$$V_m = \frac{\alpha^2}{\tau}\left[\frac{1}{n+1}\frac{\Gamma(n+2)}{\Gamma(n+\frac{5}{2})} - \frac{(n+1)^2}{4(n+2)}\frac{\Gamma(n+\frac{5}{2})}{\Gamma(n+2)}\frac{\mathrm{Pe}\tau}{\alpha^2}\right]^2. \tag{S39}$$

This formula is valid when the term in square brackets is non-negative (the term in square bracket being negative corresponds to insufficient heat production or too-rapid advection to cause widening of the ice stream).

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